The Mathematics Educator
2019 Vol. 28, No. 2, 33–51

Generalization, Assimilation, and Accommodation

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Generalization is critical to mathematical thought and to learning mathematics. However, students at all levels struggle to generalize. In this paper, I present a theoretical analysis connecting Piaget's assimilation and accommodation constructs to Harel and Tall's (1991) framework for generalization in advanced mathematics. I offer a theoretical argument and empirical examples of students generalizing graphing from R^2 to R^3. The work presented here contributes to the field by (a) drawing attention to particular cognitive activities that underpin generalization, (b) explaining empirical findings (my own and others') occurring as a result of particular cognitive activities, and (c) providing implications for influencing student cognition in the classroom.

Generalization is a key component of mathematics. Mathematicians seek general formulae, kindergarteners generalize when they seek the next shape in a pattern, and undergraduates generalize ideas from $\mathbb{R}^2$ to $\mathbb{R}^3$ to $\mathbb{R}^n$. Because generalization is critical to mathematical thought and to learning mathematics, research that investigates how people generalize supports student learning at all levels.

Descriptions of how people generalize often come in the form of frameworks (e.g., Ellis, 2007; Ellis, Tillema, Lockwood, & Moore, 2017; Harel & Tall, 1991). Frameworks provide language to describe and account for qualitative differences in students' thinking and activity. They can also reveal ways students arrive at the same generalization via different means. Frameworks support theory building by providing language to explain and predict phenomena. Because students at all levels often struggle to generalize (Ellis, 2007), developing frameworks and theory about the cognitive activities involved in generalization can inform instruction for students across multiple mathematical levels and topics.

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To that end, this paper connects a particular generalization framework (Harel & Tall, 1991) to Piagetian learning theory. The paper is a theoretical analysis, grounded in empirical findings (my own and others') about students' generalization of function and graphing from the single-variable to multivariable context. I focus on generalization in this context because it is a critical yet difficult transition for calculus students (e.g., Dorko, 2016, 2017; Dorko & Lockwood, 2016; Dorko & Weber, 2014; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010).

The work presented here contributes to the field by (a) drawing attention to particular cognitive activities that underpin generalization, (b) explaining empirical findings (my own and others') as occurring as a result of particular cognitive activities, and (c) providing implications for influencing student cognition in the classroom. The paper is organized in the following way. First, I review Harel and Tall's (1991) generalization framework and offer a theoretical analysis of how it aligns with assimilation and accommodation (Gallagher & Reid, 2002; Piaget, 1980; von Glasersfeld, 1995). Then, I offer empirical evidence of the theoretical links I propose. I present evidence both from my own data and from the larger collection of empirical findings about student generalization of function and graphing from the single- to multivariable setting. I conclude by discussing how Harel and Tall's framework as coupled with assimilation and accommodation can be useful to instructors and researchers alike.

Theoretical Analysis

In 1991, Harel and Tall proposed a framework intended to "shed some light on the different qualities of generalization in advanced mathematics. . . [and] suggest pedagogical principles designed to assist students' comprehension of advanced mathematical concepts" (p. 38). They defined generalization as "the process of applying a given argument in a broader context" (Harel & Tall, 1991, p. 38) and proposed three types of generalization:

- *Expansive generalization* "occurs when the subject expands the applicability range of an existing schema without reconstructing it" (Harel & Tall, 1991, p. 38). The original schema is "included directly as [a] special case in the final schema" (p. 38).
- *Reconstructive generalization* "occurs when the subject reconstructs an existing schema to widen its applicability range" (Harel & Tall, 1991, p. 38). The original schema "is changed and enriched before being encompassed in the more general schema" (p. 38).
- *Disjunctive generalization*[1] "occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available" (Harel & Tall, 1991, p. 38).

Harel and Tall (1991) did not define what they mean by "schema." I interpret their use of schema as consistent with von Glasersfeld's (1995) interpretation of scheme, which is based on Piaget's work. Von Glasersfeld (1995) described Piaget's construct of scheme as consisting of "(1) recognition of a certain situation; (2) a specific activity associated with that situation; and (3) the expectation that the activity produces a certain previously experienced result" (p. 65; see also Figure 1).

1		2		3
Perceived situation	→	Activity	→	Beneficial or expected result

Figure 1. Three parts of a scheme (adapted from von Glasersfeld, 1995, p. 65)

[1] Jones and Dorko (2015) argued disjunctive generalization is not a form of generalization. Because the new conception is separate from the original conception, the student is not generalizing. They suggested renaming this category as *disjunctive understanding*. Because I did not observe instances of disjunctive understandings in the data set reported on in this paper, I was unable to tease out any relationships between this category and assimilation and accommodation; whether such relationships exist is an avenue for future work.

I argue that a more explicit coupling between Piagetian learning theory and Harel and Tall's categories allows us to understand the cognitive basis for expansive and reconstructive generalization. Specifically, I propose students engage in expansive generalization by assimilating a new context to an existing scheme and engage in reconstructive generalization by accommodating an existing scheme.

Assimilation is "the integration of new objects or new situations and events into previous schemes" (Piaget, 1980, p. 164 as cited in Steffe, 1991, p. 192). It occurs "when a cognizing organism fits an experience into a conceptual structure it already has" or "treat[s] new material as an instance of something known" (von Glasersfeld, 1995, p. 62). *Generalizing assimilation* "describes the situation in which a scheme is extended to an ever-increasing number of objects" (Gallagher & Reid, 2002, p. 66). I interpret Harel and Tall's definition of expansive generalization as compatible with Piaget's definition of generalizing assimilation, which is why I argue that assimilation is the cognitive mechanism for expansive generalization.

Accommodation is a modification of a scheme that occurs due to perturbation (von Glasersfeld, 1995). When an individual's attempt to assimilate a situation to a scheme has an unexpected result, the individual experiences perturbation and disequilibrium. The individual may seek to re-attain equilibrium in several ways. The individual might modify the activity of the scheme, modify the activation criteria for the scheme by forming a recognition pattern including the new characteristic, or modify the expected result (von Glasersfeld, 1995; see Figure 1). Accommodation as modifying a scheme is consistent with Harel and Tall's (1991) definition of reconstructive generalization as "changing and enriching" or "reconstructing" (p. 38) a schema, suggesting accommodation as a cognitive mechanism for reconstructive generalization. (Note also it is possible that an individual experiences perturbation but is unable to accommodate a scheme; theoretically, this might result in a disjunctive generalization.[2])

Per developing theory, the proposed alignment between expansive generalization and assimilation and between reconstructive generalization and accommodation explains something Harel and Tall's (1991) framework did not: why a student might engage in one type of generalization instead of another. Harel and Tall did not explain on a cognitive level *why* one might reconstruct a scheme. Piagetian learning theory offers an explanation: Students engage in reconstructive generalization because they attempt to assimilate an experience to a scheme, experience perturbation and disequilibrium, and modify the scheme to re-equilibrate.

In the next section, I provide some empirical data as evidence of the value of this theoretical work.

Empirical Evidence

The data excerpts in this paper come from a longitudinal study of calculus students' generalization of the function concept from single- to multivariable settings (Dorko, 2017). I conducted four task-based clinical interviews (Hunting, 1997) with each of five students over the span of their differential, integral, and multivariable calculus courses. The total interview time ranged from 4.25 to 5.67 hours per student. Asking students about multivariable topics before instruction about those topics was a key part of the study design because it afforded insight into students' initial sense-making. The tasks discussed in this paper and the course in which students were enrolled at the time of the study are shown in Table 1. In all graphing questions, students were provided with an image of $\mathbb{R}^3$ coordinate axes.

In this paper, I focus the discussion on Wendy,[3] whose responses to these tasks provide an example of how the five

[2] Von Glasersfeld (1995) also wrote, "if an unexpected result happens to be a desirable one, the added condition may serve to separate a new scheme from the old. In this case, the new condition will be central in the recognition pattern of the new scheme" (p. 66). I interpret this as a potential mechanism for disjunctive generalization/understanding.

[3] Gender-preserving pseudonym.

students assimilated and accommodated their function and graphing schemes so as to generalize them from the single- to multivariable case. Wendy was one of four students who drew correct graphs, and I focus on her because she was the most articulate in describing her thinking.

Table 1.
Interview tasks by course

	Task	Course
1.	What does $f(x)$ mean to you?	Differential calculus
2.	What do you think $f(x, y)$ means?	Differential calculus
3.	Graph $y = x$ in $\mathbb{R}^3$.	Multivariable calculus
4.	Graph $y = 2x + 1$ in $\mathbb{R}^3$.	Multivariable calculus
5.	Graph $z = 4$ in $\mathbb{R}^3$.	Multivariable calculus

Assimilation and Expansive Generalization: An Example

I observed examples of assimilation and expansive generalization when I asked differential calculus students what they thought $f(x, y)$ meant. Wendy's thinking is representative of how students tended to answer this question.

Excerpt 1. Assimilating $f(x, y)$ to a scheme for $f(x)$ notation.

Wendy: So I know what $f(x)$ means. That means you're, you're using x to solve for y, so it would be like $2x + 4$ and then whatever you get when you plug in x is your y coordinate. But it looks like in this one you would have something like $f(x, y) = x^2 + y^2$. Because I know that when x is in the parentheses here it's what you're putting in for the equation. So if you're putting x into the equation when there's just this, if there's y too, then you would put y into the equation.

I interpret "so I know what $f(x)$ means" as evidence that seeing $f(x, y)$ activated Wendy's existing schema for function notation. I take "I know that when x is in the parentheses here it's what you're putting in for the equation" as evidence that Wendy's schema entailed $f(x)$ as relating to an expression with x's. She generalized that $f(x, y)$ would indicate an equation with x's and y's, reasoning "if there's y [in the parentheses] too, then you would put *y* in the equation." As such, Wendy fit $f(x, y)$ into a conceptual structure she already had: she assimilated it, which afforded an expansive generalization. Wendy seemed to apply the notion of the function's argument as specifying the variables in the expression to the unfamiliar $f(x, y)$ case. This follows Harel and Tall's (1991) definition of generalization as "applying a given argument in a broader context" (p. 38). It is expansive generalization because Wendy expanded the applicability range of an existing schema. I propose that Wendy's focus on the variables "in the parentheses" indicates that her function notation schema entailed a $f(\square)$ template in which $\square$ specified what symbols would appear on the right side of the equation. Wendy's generalization that $f(x, y)$ specified an equation with x's and y's was expansive because she interpreted $f(x, y)$ within her existing meaning for $f(\square)$. In short, I argue Wendy was able to generalize expansively because she treated $f(x, y)$ as something known (assimilated it to a scheme for $f(x)$).

In the next section, I provide an example of a reconstructive generalization and how accommodation afforded reconstructing a scheme.

Accommodation and reconstructive generalization: An example

I observed reconstructive generalizations when I asked students to graph the following three equations in $\mathbb{R}^3$: $y = x$, $y = 2x + 1$, and $z = 4$. Four students initially drew $y = x$, $y = 2x + 1$ as lines and $z = 4$ as a plane. These students then reconsidered their responses and drew all three equations as planes. The fifth student drew all three equations as lines and resisted my attempts to perturb this reasoning (Dorko, 2016, 2017).

Wendy's thinking provides an example of students who drew lines for the $y =$ equations before engaging in a reconstructive generalization and determining the graphs would be planes. As Excerpts 2, 3, and 4 show, Wendy appeared to have two schemes for linear equations in $\mathbb{R}^2$: a scheme for $y = mx + b$ ($m \neq 0$) and a scheme for $y = mx + b$ ($m = 0$), which I will refer to as $y = b$. She assimilated $y = x$ and $y = 2x + 1$ in

$\mathbb{R}^3$ to the former and assimilated $z = 4$ to the latter. Wendy drew $y = x$ and $y = 2x + 1$ as lines on the xy plane (Excerpt 2 and 3). In contrast, she drew $z = 4$ as a plane (Excerpt 4). Unprompted by me, Wendy compared her graphs and was puzzled by the fact that two graphs were lines and one was a plane. This comparison served as a perturbation that caused Wendy to reconstruct her scheme.

In Excerpts 2 and 3, I provide what I take as evidence of Wendy's assimilating $y = x$ and $y = 2x + 1$ in $\mathbb{R}^3$ to a scheme for $y = mx + b$ ($m \neq 0$) in $\mathbb{R}^2$.

Excerpt 2. Assimilating $y = x$ in $\mathbb{R}^3$ to a scheme for $y = x$ in $\mathbb{R}^2$.

Wendy: So if you just plug in values for *x* and then pull out values for *y*, you're gonna get like 0, 0, 1, 1, 2, 2 [*plots these on the xy plane as she says them*] and then it's just going to continue being a straight line out like this. You could choose any *x* value, really. I chose like 1. So if *x* is 1, then *y* is equal to *x*, so that's also 1.

Int.[4]: Can you label some of the coordinates that you plotted?

Wendy: Okay, so this is going to be like 1, 1, 0 and then 2, 2, 0.

Int.: Why do we get a line here?

Wendy: The way I think of it is it's just like having a 2D graph and plotting $y = x$ and that'll give you a line, you're just taking it and adding and then ignoring the *z* component. If $y = x$, you can just always assume that *z* is 0.

Excerpt 3. Assimilating $y = 2x + 1$ in $\mathbb{R}^3$ to a scheme for $y = 2x + 1$ in $\mathbb{R}^2$.

Wendy: I'm thinking that it will be like the same kind of concept where we're just ignoring *z* so you can say like $+0z$ here and that will give you the same equation [*writes* $y = 2x + 1 + 0z$]. So if you went $2x + 1$ that would be 0, 1 and then 1, 3. Basically you would just take the same line that you would have with your *x* and *y*.

Int.: And do we get a line there?

[4] "Int." is an abbreviation for 'interviewer.'

Wendy: Yeah, that's a line. Like I said we're ignoring the z component, but you can think of it as there, you're just, have it, 0 set to it.

I argue that Wendy assimilated these equations to a scheme for $y = mx + b$ ($m \neq 0$) in $\mathbb{R}^2$. As evidence, Wendy talked about the coordinate points as (x, y) tuples (e.g., "0, 0, 1, 1, 2, 2") as she was plotting the points (Excerpt 2). Though she described the points as (x, y, z) tuples when asked to identify points, I posit that her thinking of the points as (x, y) tuples during the act of graphing indicates that she had assimilated the question about creating a graph in $\mathbb{R}^3$ to a schema for graphing in $\mathbb{R}^2$. Wendy's statement that she saw $y = x$ in $\mathbb{R}^3$ as "just like having a 2D graph and plotting $y = x$" supports this inference.

I argue Wendy's treatment of z facilitated her assimilation. We know Wendy considered z because she said in both graphs that she was "ignoring z" (Excerpt 2 and 3) or setting it to 0 (Excerpt 3)[5]. Further, when asked what points she had plotted on her $y = x$ graph, Wendy gave (x, y, z) tuples. Wendy's statements about z provided evidence that she (a) explicitly considered z and (b) treated it in a way that allowed her to assimilate the $y = x$ and $y = 2x + 1$ in $\mathbb{R}^3$ tasks to a scheme from $\mathbb{R}^2$. This is in accordance with assimilation as "reduc[ing] new experiences to already existing sensorimotor or conceptual structures" (von Glasersfeld, 1995, p. 63).

In contrast, I take Excerpt 4 as evidence of Wendy assimilating $z = 4$ to a different scheme. Specifically, she appeared to have a scheme for graphing $y = b$ in $\mathbb{R}^2$ and expansively generalized it to the case of $z = 4$ in $\mathbb{R}^3$.

Excerpt 4. Assimilating $z = 4$ in $\mathbb{R}^3$ to a scheme for $y = b$ in $\mathbb{R}^2$.

Wendy: I'm thinking that whenever, no matter what x and y equal, z is always going to equal 4. So you get a plane here at 4. That's a really bad drawing of it, but, no matter

[5] For Wendy, writing +0z meant that she was setting z to 0 (Excerpt 2). This contrasts a normative interpretation of $y = 2x + 1 + 0z$, in which one sees z as varying.

what these [*gestures to x axis*] equal, you're always just going to get 4.

Int.: Can you tell me a little bit more about the "no matter what these equal"?

Wendy: So if you're graphing, so $z = 4$, it's like saying $y = 4$ on a normal graph you get a line at y, or 4. You just get that [*sketches* $y = 4$ *in* $\mathbb{R}^2$]. Because it doesn't matter what x equals. So here I'm kind of thinking that it's the same concept, that no matter what y or x equals, z is always going to equal 4.

Int.: Do you, as you graphed that $z = 4$, so you pretty immediately said oh, this is a plane. Did you think about this y and x graph? [*points to Wendy's graph of* $y = 4$ *in* $\mathbb{R}^2$].

Wendy: I basically, I took the concept of it and applied it.

Int.: And what's the concept of it?

Wendy: Yeah, the concept of it is like I said even though there's no x in this equation, like we always know that y is going to be equal to 4 so it really doesn't matter what x is, so that's why there's no x in the equation.

Int.: How come $z = 4$ isn't just a line?

Wendy: Because you're in 3D, so if say like x was 1 and y was 2, you're always, z is going to equal 4.

I take Wendy's comment "it's the same concept" as evidence that she assimilated $z = 4$ to an already-existing scheme. What Wendy appeared to see as the "same concept" was that $y = 4$ in $\mathbb{R}^2$ "[no] matter what x equals," so z would equal 4 in $\mathbb{R}^3$ "no matter what y or x equals." Wendy argued that $z = 4$ was a plane using the example of (1, 2, 4) as a point on the graph.

I contend Wendy assimilated $z = 4$ to a different scheme than the one to which she assimilated $y = x$ and $y = 2x + 1$. That is, Wendy appeared to have a scheme for constant functions in $\mathbb{R}^2$, an element of which was that x was free. She expansively generalized this to the $\mathbb{R}^3$ case by viewing x and y as free. In contrast, she appeared to have a scheme for non-constant linear functions, an element of which was that such functions' graphs are lines. Wendy expanded this scheme to the $\mathbb{R}^3$ case by "ignoring z" or, equivalently in her mind, "set it to 0."

Wendy's assimilations to two different schemes resulted in two different graphs, triggering a perturbation that subsequently caused Wendy to reconstruct her scheme for non-constant linear equations in $\mathbb{R}^3$ (Excerpt 5).

Excerpt 5. Perturbation.

Int.: So it's interesting to me that –
Wendy: It's interesting to me too.
Int.: What's interesting to you too?
Wendy: That I think of that [$z = 4$] like that, and then the other ones [$y = x$ and $y = 2x + 1$] I don't think of like that. So if I, if I applied what I did in [$z = 4$] to [$y = x$ and $y = 2x + 1$] I would get planes again, which would look like this . . . because *y* is going to equal *x*. I feel like I'm confusing myself.

Wendy then compared her work on the three graphs, which led her to reconstruct her notion of a free variable (Excerpt 6).

Excerpt 6. Accommodation.

Int.: Okay, so do you want to look at these again? [*puts* $y = x$ *and* $y = 2x + 1$ *graphs in front of Wendy*]
Wendy: So if I think about it like this [*points to* $z = 4$ *graph*], so if I thought of this [$z = 4$] like I think of this [*points to* $y = 2x + 1$], then this [$z = 4$] would just be a point.
Int.: Can you say that sentence [again]? The word *this* gets hard when I do the audio, when I transcribe it.
Wendy: Okay so on the previous ones I was thinking of, I was thinking of this [$y = x$] as—this—the $y = x$ as just like $y = x$ and then I was thinking of it as $+ 0z$. And so out of that you get a line. But instead of thinking of this $+ 0y + 0x$, I thought of it as more of the $y = 4$. That no matter what the, no matter what the *y* and *x* values are here, the *z* is always going to equal 4—so if I, if I applied what I did in [$z = 4$] to [$y = x$ and $y = 2x + 1$] I would get planes again, which would look like this [*sketches a plane*]. Because *y* is going to equal *x*. I feel like I'm confusing myself.

Int.: So, so do you think $y = x$ in $\mathbb{R}^3$ is a plane or a line?

Wendy: My initial thought was that it was a line, but now I'm unsure—My initial thought process of it's a line is because I was thinking that you didn't change this x and y coordinate, you just laid it flat, and that is the only thing you did to make it 3D here. And so you could just graph it in 2D and then just lay it flat and put a z axis in it and that wouldn't change the $y = x$. But that was if I was thinking + 0z which there isn't a + 0z. So I think that no matter what z is, y is always going to equal x. So whatever x and y are, you're going to have that plane.

I interpret the change in Wendy's graph from a line to a plane as occurring as a result of the following. Wendy's statement that she found it "interesting" that she had drawn a line for two of the graphs and a plane for the third suggests that she expected the graphs to look similar. The unexpected results (the graphs did not look similar) caused a perturbation. Wendy sought to re-equilibrate (remove the perturbation) by comparing how she approached the $y =$ equations and the $z = 4$ equation. In doing so, she noticed that in the $y =$ equations she had assumed $z = 0$, while in the $z = 4$ equation she had assumed x and y could take on any value. Wendy accommodated her scheme for $y = x$ as entailing that z was equal to 0 to $y = x$, $z \in \mathbb{R}$. The original schema was reconstructed because she "changed and enriched it" (Harel & Tall, 1991, p. 1) by viewing z as a free variable. This allowed for a more general schema in which Wendy could see the three equations $y = x$, $y = 2x + 1$, and $z = 4$ as having a free variable(s) and as planes.

I offer the above analysis to establish that assimilation and accommodation can explain students' cognitive activity while generalizing mathematical ideas. In the next section, I offer additional examples of students' generalizing from single- to multivariable functions and discuss how Harel and Tall's framework, as more explicitly connected to assimilation and accommodation, has explanatory power for other researchers' findings about generalization in undergraduate mathematics contexts.

Assimilation, Accommodation, Generalization, and the Larger Literature Base

In this section, I draw on examples from the literature of how students have generalized the concept of function and of graphing multivariable functions. I do so to illustrate the explanatory power of the theoretical analysis presented above.

Function machine

Thinking about functions in terms of inputs and outputs helps students generalize the function notion from the single- to multivariable context (Dorko & Weber, 2014; Kabael, 2011). Tall, McGowen, and DeMarois (2000) suggested that the function machine model is powerful because it provides a *cognitive root*, “a concept that is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence [that] allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction [and] contains the possibility of long-term meaning in later developments” (p. 3). The phrase “cognitive expansion rather than significant cognitive reconstruction” is suggestive of expansive generalization. I argue the function machine provides a scheme to which students can assimilate, and that assimilation is the cognitive mechanism that allows students to engage in expansive generalization. Empirical findings bear this out. For example, Dorko and Weber (2014) described how some students generalized domain and range in terms of inputs and outputs. They provided an example of a student answering “What are the domain and range of $f(x, y) = x^2 + y^2$?” by saying

> Domain is your input values . . . The range is your output . . . There would be two different domains. You have your x input and your y input. Your x domain and your y domain give you a range of a different variable. It’s the range of z or $f(x, y)$. (Dorko & Weber, 2014, p. 8)

Dorko and Weber (2014) described this generalization as the student *extending* (Ellis, 2007) a meaning that domain corresponded to inputs and range corresponded to outputs. I argue this student engaged in expansive generalization because he assimilated the multivariable function to his already existing scheme of function as having inputs and outputs.

Kabael (2011) studied students' generalization of function when taught multivariable functions using the input-output model and gave examples of students engaging in a similar extension (p. 492). I theorize Kabael's students could engage in expansive generalization because the function machine allowed them to assimilate multivariable functions to their input-output scheme of single-variable functions.

In summary, empirical findings about students' generalization of function, characterized with different generalization frameworks, indicate students often extended their meaning for function, using function machine as a cognitive root. I argue these are expansive generalizations in Harel and Tall's language, and more importantly, no matter what word we use, these generalizations are afforded by assimilation. An instructional implication is that developing cognitive roots for other mathematical topics may support students' generalizations of those ideas.

Graphing

Researchers have observed student difficulties with graphing in $\mathbb{R}^3$, particularly equations with free variables (Dorko & Lockwood, 2016; Martínez-Planell & Trigueros, 2012; Trigueros & Martínez-Planell, 2010). For example, students may draw $f(x, y) = x^2 + y^2$ as a cylinder or a sphere because they are accustomed to $x^2 + y^2$ representing a circle in $\mathbb{R}^2$ (Martínez-Planell & Gaisman, 2013). As another example, students may draw $f(x, y) = x^2$ as a parabola (instead of a parabolic surface) in $\mathbb{R}^3$ (Martínez-Planell & Gaisman, 2013). I posit students assimilate $f(x, y) = x^2 + y^2$ and $f(x, y) = x^2$ to their $\mathbb{R}^2$ schemes for the expressions $x^2 + y^2$ and x^2 respectively. In support of this, Moore, Liss, Silverman, Paoletti, LaForest, and Musgrave (2013) documented that students often created graphs based on *static shape thinking*, or "conceiving of graphs as pictorial objects" (Moore et al., 2013, p. 441; Moore & Thompson, 2015). That is, a possible explanation for students' graphs of the aforementioned equations is that they assimilated multivariable functions' equations to their schemes for the shapes of graphs in $\mathbb{R}^2$, which allowed them to expansively generalize by creating similar shapes on $\mathbb{R}^3$ coordinate systems. Wendy's expansive generalization of non-constant linear functions may be another example of a student engaging in static shape thinking.

Harel and Tall's framework as coupled with assimilation and accommodation provides an explanation for students' static shape thinking in $\mathbb{R}^3$, and moreover, it offers implications for how to support students in creating correct graphs. If students are assimilating when we would like them to accommodate their graphing schemes, we can do a better job of teaching graphing by providing learning opportunities intended to induce perturbation and support students in reconstructing their graphing schemes (Steffe & Wiegel, 1992). As Excerpt 6 described, comparing two graphs was sufficient to engage Wendy in a reconstructive generalization. Future research should focus on what sorts of learning opportunities can induce perturbation and subsequent reconstruction (Moore, Stevens, Paoletti, Hobson, & Liang, 2019; Steffe, 1991).

Conclusion

I have argued that assimilation and accommodation align with Harel and Tall's (1991) expansive and reconstructive generalization categories, respectively. Connecting frameworks and theories helps us better understand phenomena of interest. There are many other generalization frameworks and definitions of generalization (Mitchelmore, 2002) and hence it is likely many more such connections can be made. For example, Ellis, Tillema, Lockwood, and Moore (2017) examined Piaget's forms of abstraction as underpinning categories of Ellis' (2007) generalization taxonomy. One might examine other generalization frameworks in terms of assimilation and accommodation. In addition to the body of work about generalization, there is also a body of work about transfer, which some researchers have connected to Piagetian learning theory (e.g., Wagner, 2010).

Finally, Harel and Tall's (1991) framework was developed with the aim of "suggest[ing] pedagogical principles designed to assist students' comprehension of advanced mathematical concepts" (p. 38). They write,

> [W]e believe that the most desirable approach to generalization is to provide experiences which lead to a meaningful understanding of the current situation, to allow the move to the more general case to occur by expansive generalization, but that there are times when the situation demands a re-construction [sic] and, in such cases, it is necessary to provide the learner with the conditions in which this reconstruction is more likely to take place. (Harel & Tall, 1991, p. 3)

Interpreted with the assimilation and accommodation connections, we can think about designing activities to support students' construction of schemes that allow students to assimilate and engage in expansive generalization. Kabael's (2011) study provides a good model. In cases where it seems students must reconstruct their schemes, empirical research such as that using the constructivist teaching experiment methodology (Steffe, 1991; Steffe & Thompson, 2000) can identify the sorts of activities that engender perturbation and whether those activities result in the desired accommodations and reconstructive generalizations.

References

Dorko, A. (2016). What do students attend to when first graphing y = 3 in R^3? In T. Fukawa-Connelly, N. Infante, M. Wawro, and S. Brown (Eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education*, (pp. 111–124). Pittsburgh, PA: West Virginia University. Retrieved from http://sigmaa.maa.org/rume/RUME19v3.pdf

Dorko, A. (2017). *Students' generalisation of function from single- to multivariable settings* (Unpublished doctoral dissertation). Oregon State

University, Corvallis, OR. Retrieved from https://ir.library.oregonstate.edu/concern/graduate_thesis_or_dissertations/9p290g05d

Dorko, A. & Lockwood, E. (2016). What do students attend to when first graphing in R3? In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 565 - 572). Tucson, AZ: The University of Arizona. Retrieved from http://www.pmena.org/pmenaproceedings/PMENA%2038%202016%20Proceedings.pdf

Dorko, A. & Weber, E. (2014). Generalising calculus ideas from two dimensions to three: How multivariable calculus students think about domain and range. *Research in Mathematics Education. 16*(3), 269–287. doi:10.1080/14794802.2014.919873

Ellis, A. B. (2007). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *Journal of the Learning Sciences, 16*(2), 221–262. doi:10.1080/10508400701193705

Ellis, A. B., Tillema, E., Lockwood, E., & Moore, K. C. (2017). Generalization across domains: The relating-forming-extending generalization framework. In E. Galindo & J. Newton (Eds.), *Proceedings of the thirty-ninth Annual Meeting of the North American Chapter of the Psychology of Mathematics Education* (pp. 667–684). Indianapolis, IN: Indiana University. Retrieved from https://files.eric.ed.gov/fulltext/ED581351.pdf

Gallagher, J. M. & Reid, D. K. (2002). *The Learning Theory of Piaget & Inhelder.* NE: Authors Choice Press.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics, 11*(1), 38–42. Available at https://www.jstor.org/stable/40248005?seq=1#metadata_info_tab_contents

Hunting, R. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior, 16*(2), 145–165.

Kabael, T. U. (2011). Generalizing single variable functions to two variable functions, function machine and APOS. *Educational Sciences: Theory and Practice, 11*(1), 484–499. Available at https://files.eric.ed.gov/fulltext/EJ919912.pdf

Martínez-Planell, R., & Gaisman, M. T. (2013). Graphs of functions of two variables: results from the design of instruction. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 663–672. doi:10.1080/0020739X.2013.780214

Martínez-Planell, R., & Gaisman, M. T. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics, 81*(3), 365–284. doi:10.1007/s10649-012-9408-8

Martínez-Planell, R., & Trigueros, M. (2012). Activity sets to help student graphing of functions of two variables. In *Proceedings of the 12th International Congress of Mathematics Education* (pp. 2759–2769). Retrieved from https://www.springer.com/us/book/9783319106854

Mitchelmore, M. C. (2002, May). *The role of abstraction and generalisation in the development of mathematical knowledge.* Paper presented at the East Asia Regional Conference on Mathematics Education (EARCOME) (2nd) and the Southeast Asian Conference on Mathematics Education (9th), Singapore. Retrieved from https://files.eric.ed.gov /fulltext/ED466962.pdf

Moore, K.C., Liss II, D.R., Silverman, J., Paoletti, T., LaForest, K.R., & Musgrave, S. (2013). Pre-service teachers' meanings and non-canonical graphs. In M., Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 441-448). Chicago, IL: University of Illinois at Chicago. Retrieved from http://www.pmena.org/pmenaproceedings/PMENA%2035%202013%20 Proceedings.pdf

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior, 56*, 100692. https://doi.org/10.1016/j.jmathb.2019.01.008

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. I. Engelke, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 782–789). Pittsburgh, PA: West Virginia University.

Piaget, J. (1980). Schemes of Action and Language Learning. In M. Piattelli-Palmarini (Ed.), *Language and Learning: The Debate Between Jean Piaget and Noam Chomsky* (pp. 23–34). Cambridge, MA: Harvard University Press.

Steffe, L. (1991). The constructivist teaching experiment: Illustrations and implications. In E. von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp. 177–194). Dordrect, the Netherlands: Springer.

Steffe, L. P. & Wiegel, H. G. (1992). On reforming practice in mathematics education. *Educational Studies in Mathematics, 23*(5), 445–465.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Mahwah, NJ: Lawrence Erlbaum Associates.

Tall, D. O., McGowen, M., & DeMarois, P. (2000). The function machine as a cognitive root for building a rich concept image of the function concept. *Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Tucson, AZ. Retrieved from https://eric.ed.gov/?q=ED446945

Thompson, P.W. (2016). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 435–461). New York, NY: Taylor & Francis.

Trigueros, M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two-variable functions. *Educational Studies in Mathematics*, *73*(1), 3–19. doi: 10.1007/s10649-009-9201-5

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.

Wagner, J. F. (2010). A transfer-in-pieces consideration of the perception of structure in the transfer of learning. *Journal of the Learning Sciences*, *19*(4), 443–479.