The Mathematics Educator
2017 Vol. 26, No. 2, 3–25

Noticing and Knowledge: Exploring Theoretical Connections between Professional Noticing and Mathematical Knowledge for Teaching

Jonathan Thomas, Cindy Jong,
Molly H. Fisher, and Edna O. Schack

For the past two decades, the development of preservice elementary teachers' mathematical knowledge and skills has been central to mathematics education research. Two frameworks that researchers have drawn upon to examine such development are mathematical knowledge for teaching and professional noticing (of children's mathematical thinking). We have identified shared theoretical space between these two frameworks, and we hypothesize that effective professional noticing occurs at the intersection of developed mathematical knowledge for teaching and a high level of responsiveness with respect to the mathematical activities of students.

The knowledge and skill of preservice teachers are subject to increasing scrutiny. Emphasis on teacher accountability for student learning outcomes necessitates the development of effective instructional tactics among all teachers including those just beginning their professional practice. In the area of mathematics education, this necessity for effectiveness is intensified in the U.S. by the still relatively widespread adoption of the Common Core State Standards for Mathematics (and concomitant assessments), which were designed to

Drs. Jonathan Thomas, Cindy Jong, and Molly Fisher *are each Associate Professors of Mathematics Education at the University of Kentucky.*

Dr. Edna Schack *is a Professor of Mathematics Education at Morehead State University.*

These authors share a research interest in responsive mathematics teaching practices including teacher noticing. Drs. Thomas, Jong, Fisher, and Schack have conducted two federally funded investigations of teacher noticing with emphasis on changes in pre-service teacher performance and relationships to adjacent constructs such as teacher knowledge and attitudes toward mathematics.

increase the rigor of students' mathematical experiences (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

The mathematical knowledge of preservice elementary teachers (PSETs) has been given considerable attention. The findings in this area demonstrate a noteworthy congruence in that many PSETs demonstrate underdeveloped, fragile, or nonexistent conceptualizations of key ideas related to the effective teaching and learning of mathematics; moreover, these conceptualizations can be remarkably resistant to experiences within the teacher education program (e.g., Adams, 1998; Ball, 1990; Conference Board of Mathematical Sciences & American Mathematical Society, 2012; Foss & Kleinsasser, 1996; Quinn, 1997; Stoddart, Connell, & Stofflett, 1993). Ultimately, this conglomeration of inquiry and Shulman's (1986) conception of pedagogical content knowledge provided the context for the construction of a specialized framework to characterize essential knowledge involved in the teaching and learning of mathematics (Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008). This framework, referred to as mathematical knowledge for teaching (MKT), has captured the attention of the field (Davis & Renert, 2014) and has influenced many subsequent inquiries (e.g., Hill et al., 2008; Russell, Anderson, Goodman, & Lovin, 2009; Shechtman, Roschelle, Haertel, & Knudsen, 2010).

PSET practice of specialized skills in the context of mathematics teaching and learning has also been extensively examined (Ball & Forzani, 2009). Recently, increasing emphasis has been placed on the professional noticing of children's mathematical thinking (Fisher et al., 2014; Jacobs, Lamb, & Philipp, 2010; Philipp, 2014; Schack, Fisher, Thomas, Eisenhardt, Tassell, & Yoder, 2013; Schack, Fisher, & Wilhelm, 2017; Sherin, Jacobs, & Philipp, 2011). Professional noticing encompasses three components that, when enacted, ostensibly result in individualized and responsive mathematics instruction. Given the sustained influence of MKT and ascendance of professional noticing as theoretical frameworks for understanding the knowledge and skills which mediate mathematics teaching and learning, it follows that inquiry into potential relationships between the two frameworks may provide fruitful terrain for a theoretical synthesis. Toward this end, we identify shared theoretical space between these two frameworks, and hypothesize that effective professional noticing occurs at the intersection of developed mathematical knowledge for teaching and a high level of responsiveness with respect to the mathematical activities of students. Although we have focused primarily on PSETs in our own empirical study (described in a following section), the theoretical orientations and subsequent conclusions are likely applicable across the entire teaching constituency.

Professional Noticing

Professional noticing is a skill teachers use to identify and act upon salient mathematical actions of children. Although professional noticing is, perhaps, not as well established in the research literature as MKT, there has been growth in the number of inquiries into professional noticing in recent years. For instance, Sherin and van Es (2009) examined video-based professional development activities and found using noticing skills positively impacted teachers' instructional practices. Additionally, many researchers have constructed professional development experiences (often involving video) designed to draw attention to the mathematical thinking of children (e.g., Carpenter, Fennema, Franke, Levi, & Empson, 1999; Schifter, Bastable, & Russell, 2000; Seago, Mumme, & Branca, 2004). Lastly, the edited volume put forth by Sherin et al. (2011) provided evidence for the value of professional noticing within systems of effective mathematics teaching and learning.

Turning to the components of professional noticing, different researchers have put forth varied constructions. Mason (2002, 2011) referred to the practice as "a collection of techniques for (a) *pre-paring* to notice in the moment … and (b) *post-paring* by reflecting on the recent past to select what you want to notice or be sensitized to" (2011, p. 37). Further, Mason described the manner in which these techniques may be organized according to *accounts of* and *accounting for.* Here, accounts of are "free of theorizing, emotional content, justification … [and] provide brief but vivid descriptions" while accounting for refers to the introduction of "theorizing, explaining, and accounting for not only what was observed but why it struck the observer sufficiently to be identified or marked" (2011, pp. 39-40). Professional noticing, as defined by Jacobs, Lamb, & Philipp (2010) refers to teachers' capacity to (a) *attend* to the student's mathematical conceptions and practices as they occur, (b) *interpret* these conceptions and practices, and (c) *decide* upon a productive instructional course of action based on this interpretation. Somewhat similarly, Jacobs et al.'s (2010) conceptualization of noticing parallels Mason's description in that attending refers to the noting of "the mathematical details in children's strategies," which relates to Mason's accounting of, while interpreting involves the coordination of the observed strategies with current theory of mathematical development, similar to Mason's accounting for. Deciding denotes the teachers' intended response based upon an interpretation which, itself, is based on evidence from attending (Jacobs et al., 2010). Enacted fluidly within a mathematical interaction between a teacher and a student, the process of attending, interpreting, and deciding should result in highly individualized and responsive instructional tactics. Jacobs et al. (2010) found preservice teachers and experienced teachers with or without extensive and focused professional development exhibit varying levels of proficiency with professional noticing. Experienced teachers with little focused professional development demonstrated professional noticing skills closer to those of preservice teachers than their counterparts who participated in focused professional development, suggesting that teaching experience alone does not necessarily result in the development of such skills.

Some researchers have appropriated the construct of professional noticing for varied inquiry, and in some instances, the presented conception appears to only emphasize the component skills of attending (Hanna, 2012) or attending and interpreting (Wickstrom, Baek, Barrett, Cullen, & Tobias, 2012). Noteworthy, here, is the component construction of noticing as well as the exclusion of instructional response or decision-making. Nevertheless, these conceptions of professional noticing seem consistent with prior studies focused on teachers' attention to children's mathematical thinking and the extent to which such attention positively impacts learning outcomes (Carpenter et al., 1999; Kersting, Givven, Sotelo, & Stigler, 2010). However, Jacobs et al. (2010) described professional noticing as a set of interrelated skills; thus, deliberate attention must be given to relating each component (including deciding) in order to affect growth in teachers' capacities in this area.

Mathematical Knowledge for Teaching

MKT refers to the amalgamated knowledge required to effectively teach mathematics. Although researchers in francophone communities have given significant attention to teachers' mathematics knowledge (Bednarz & Proulx, 2009), the work of Ball and her colleagues significantly increased the prominence of MKT as a framework for categorizing and describing the different knowledge domains in mathematics teaching. MKT provides a structured perspective of subject matter and pedagogical content knowledge related to mathematics teaching (see Figure 1).

In Figure 1, each domain refers to knowledge necessary for mathematics teaching. While some of these domains are, perhaps, somewhat self-explanatory (e.g., common content knowledge, knowledge of curriculum), others might benefit from additional explanation. For example, specialized content knowledge refers to the "mathematical knowledge and skill unique to teaching" and is "not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400). One instance of specialized content knowledge could involve considering the multiplication strategies constructed by students, not just for correctness, but actually being able to monitor and understand the supporting mathematics (Harkness & Thomas, 2008). Similarly, knowledge at the mathematical horizon refers to an "awareness of how mathematical topics are related over the span of mathematics included in the curriculum. . . . It also includes the vision useful in seeing connections to much later mathematical ideas" (Ball et al., 2008, p. 403). Indeed, each

MKT domain is a specified knowledge type identified as essential for mathematics teaching.

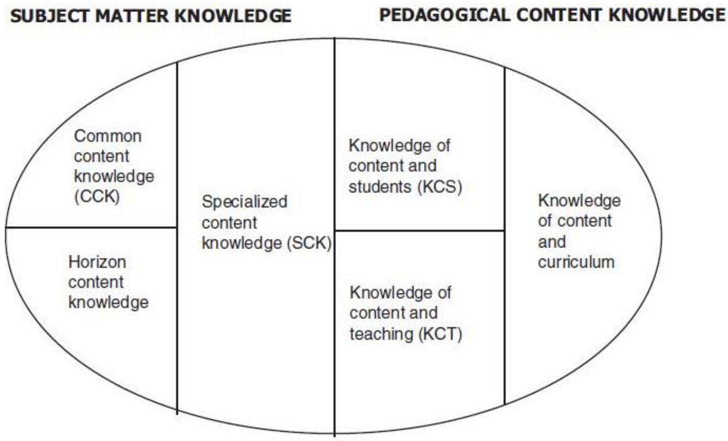

Figure 1. MKT Domains (Ball et al., 2008).

Potential Connections between Professional Noticing and MKT

There is potential for connections between professional noticing and other perspectives and constructs (i.e., attitudes and beliefs toward mathematics); however, our primary aim of this essay is to examine a possible relationship between professional noticing and MKT. Prediger, Bikner-Ahsbahs, and Arzarello (2008) described such efforts to link theoretical frameworks as a networking strategy. They wrote, "Networking strategies are those connecting strategies that respect on the one hand the pluralism and/or modularity of autonomous theoretical approaches but are on the other hand concerned with reducing the unconnected multiplicity of theoretical approaches in the scientific discipline" (p. 170). We recognize the benefit of distinct frameworks that encompass responsive teaching practice (professional noticing) and requisite knowledge for mathematics teaching (MKT); however, we find considerable merit in this potential reduction of theoretical isolation. Indeed, Hill, Ball, and Schilling (2008) explicitly call for a theoretical space "that would help us examine the nature of teachers' mathematical-pedagogical reasoning about students" (p. 396).

MKT was a natural outgrowth of Shulman's (1986) identification of a research gap with respect to content and pedagogy. Shulman hypothesized such research would deliberately inform discussions centered on the professional knowledge base of teachers. Thus, the formulation of MKT was an attempt to define and describe such knowledge in the area of mathematics teaching. Conversely, professional noticing has been linked to Dewey's (1904) descriptions of outer and inner attention (Erickson, 2011). Dewey described outer attention as the easily observed behavior cues (e.g., sitting still, talking to a neighbor, etc.) while inner attention referred to internal interests and thoughts of the student. One may draw clear parallels to the constructs of attending and interpreting and the extent to which these component skills may more effectively mediate instructional experiences (Jacobs et al., 2010). Ultimately, though, the identification and elaboration of professional noticing is organized around the practice of teaching. Therefore, fundamental to this juxtaposition of MKT and professional noticing is the relationship between knowledge and practice.

Certainly, intersecting knowledge of pedagogy and content is a prerequisite for meaningful practice, and such knowledge has been linked to improved student learning outcomes (Hill, Rowan, & Ball, 2005). While researchers have examined a wide range of teaching practices (Brophy & Good, 1986; Westwood, 1996), germane to this essay are those organized around instructional responsiveness. Sometimes referred to as *adaptive instruction*, this characterization of teaching is defined as "instruction geared to the characteristics and needs of individual students" (Westwood, 1996, p. 74). Such instruction has been linked to improved student learning outcomes (Waxman, Wang, Anderson, & Walberg, 1985).

Regarding preservice teachers' connections between knowledge and practice, Tsamir (2005) examined prospective teachers' familiarity with the "intuitive rules theory" (common, but incorrect, solution strategies based on intuition) and the positive impact this has on their subject-matter knowledge and pedagogical-content knowledge. Tsamir argued that this familiarity allows PSETs to better analyze their own thinking, the thinking of children, and the tasks proffered for studying specific topics.

Building upon these previous researchers' connections between knowledge and practice, the frameworks of professional noticing and MKT may be similarly interconnected. The manner in which these two frameworks have been constructed suggests a potential relationship despite their ostensibly different nature. Indeed, professional noticing allows for theoretically locating and analyzing responsive instructional practices while MKT provides a framework for considering and investigating the varied knowledge-types required for rich mathematics teaching. Nevertheless, these theoretical lenses appear to solidly intersect in some of their respective constructions. Consider the description of one MKT component, horizon content knowledge, put forth by Ball and Bass (2009):

> We define horizon knowledge as an awareness–more as an experienced and appreciative tourist than as a tour guide–of the large mathematical landscape in which the present experience and instruction is situated. . . . It is a kind of knowledge that can guide the following kinds of teaching responsibilities and acts: Making judgments about mathematical importance; Hearing mathematical significance in what students are saying; Highlighting and underscoring key points; Anticipating and making connections; *Noticing and evaluating* [emphasis added] mathematical opportunities. (p. 6)

The authors described the manner in which certain types of specialized knowledge may inform specific teaching practices, and for each of the presented examples of "responsibilities and acts" there are clear connections to professional noticing. Interestingly, Ball and Bass (2009) specifically used the term "noticing" in one of these examples.

Further, in their description of the enactment of MKT, Adler and Davis (2006) wrote, "a second problem for the teacher is that he or she would need to interpret the specific mathematical thinking and reasoning in which each learner has engaged. . . . the teacher will also need to figure out how to engage these interpretations in the classroom–how to mediate between them and the [intended] mathematical notions" (p. 274). Adler and Davis referred to such applications of MKT as *unpacking* and *decompressing*. Building on the idea of acting upon (e.g., interpreting, mediating) students' mathematical thinking, Chick (2009) posited that "how teachers choose and use examples in the classroom should … provide insights into their knowledge of mathematics for teaching. Moreover, this area is worth examining because examples are at the critical nexus between pedagogy and content" (p. 26). From Chick's perspective, the use of mathematical examples signifies both pedagogical unpacking and decompressing as well as some knowledge of the general mathematical idea embodied within the example.

Delving more deeply into the component skills of noticing, in their portrayal of interpreting and deciding, Jacobs et al., (2010) offered language that suggested strong connections to MKT. Consider the following quotations on interpreting and deciding, respectively:

> On the basis of a single problem, we do not expect a teacher to construct a complete picture of a child's understandings, but we are interested in the extent to which the *teacher's reasoning is consistent with both the details of the specific child's strategies and the research on children's mathematical development* [emphasis added] (Jacobs et al., 2010, p. 172)

> We are interested in the extent to which teachers use what they have learned about the children's understandings from the specific situation and whether their *reasoning is consistent with the research on children's mathematical development* [emphasis added] (Jacobs et al., 2010, p. 173).

In the enactment of the component skill of interpreting, Jacobs et al.'s (2010) reference to the "child's strategies" and "research on children's mathematical development" imply a strong connection to the pedagogical content knowledge portions of the MKT framework (e.g., knowledge of content and students; knowledge of content and teaching). Such connections appear to persist through the component skill of deciding as they invoke knowledge of "research on children's mathematical development" which strongly implies portions of MKT (pp. 172–173) as well as, perhaps, the construct of hypothetical learning trajectories. Further, Towers and Martin's (2009) exploration of improvisation (i.e., willingness to modify actions in response to external cues) as it relates to preservice teachers' development of MKT provides additional elaboration on the relationship between knowledge (as it develops) and classroom practices. Specifically, the analytical incorporation of improvisational theory demonstrates how teaching and learning practices "can reformulate unpacked mathematics knowledge into knowledge for mathematics teaching" (Towers & Martin, 2009, p. 48).

From the perspectives of the authors of MKT as well as the authors of professional noticing, knowledge and practice are necessarily conjoined in the process of meaningful mathematics teaching and learning. Davis and Renert's (2014) formulation of mathematics for teaching (M_4T) both extends and subsumes previous attempts to define the construct, resulting in a definition that seems to merge mathematics for teaching knowledge with professional noticing practice. Davis and Renert write, "M_4T is a way of being with mathematics knowledge that enables a teacher to structure learning situations, interpret student actions mindfully, and respond flexibly, in ways that enable learners to extend understandings and expand the range of their interpretive possibilities through access to powerful connections and appropriate practice" (p. 4).

Moreover, related inquiries support such a nexus of knowledge and noticing. Given this occasion for theoretical connection, there is cause to more closely examine the extent to which they both initially develop among educators (PSETs, in this instance), and the extent to which such developing skills and knowledge may be related.

Elaboration upon the Theoretical Landscape

Illustrated in the extant literature, there are clear overlaps in construction between the two theoretical frameworks (Ball & Bass, 2009; Davis & Renert, 2014; Jacobs et al., 2010; Prediger, 2010; Tsamir, 2005). Making appropriate interpretations of children's mathematical thinking requires specialized mathematical knowledge, as do the instructional decisions that follow such interpretations. However, we contend such specialized mathematical knowledge is necessary but ultimately not sufficient for effective professional noticing.

Recalling the interrelated nature of the component skills of professional noticing (attending, interpreting, deciding), theoretically, the initial skill of attending to the mathematical activity of the student necessarily frames the subsequent skills (interpreting, deciding) as responsive to such activity. For example, attending to the nuanced features of a child's particular counting strategy leads to a specific interpretation of mathematical understanding with respect to that child—e.g., "I believe the child is operating with a perceptual (concrete) counting scheme" (Steffe, 1992). Subsequently, this interpretation, ideally, leads to an instructional decision tailored to advance that child's mathematical thinking—e.g., "I will cover the materials to help the student generate quantitative imagery" (Thomas & Tabor, 2012). Certainly, specialized knowledge is leveraged to professionally notice in this instance, but there is also considerable responsiveness inherent in the process. Thus, we contend that one productive lens for considering the practical outcomes of professional noticing is from the perspective of responsiveness. In this instance, responsiveness may be considered a broad manifestation of the coordinated component skills of professional noticing. Moreover, this characterization of responsiveness is consistent with Westwood's (1996) description of adaptive instruction. Toward this end, we have hypothesized certain practical outcomes that may occur when juxtaposing such responsiveness with MKT (see Figure 2).

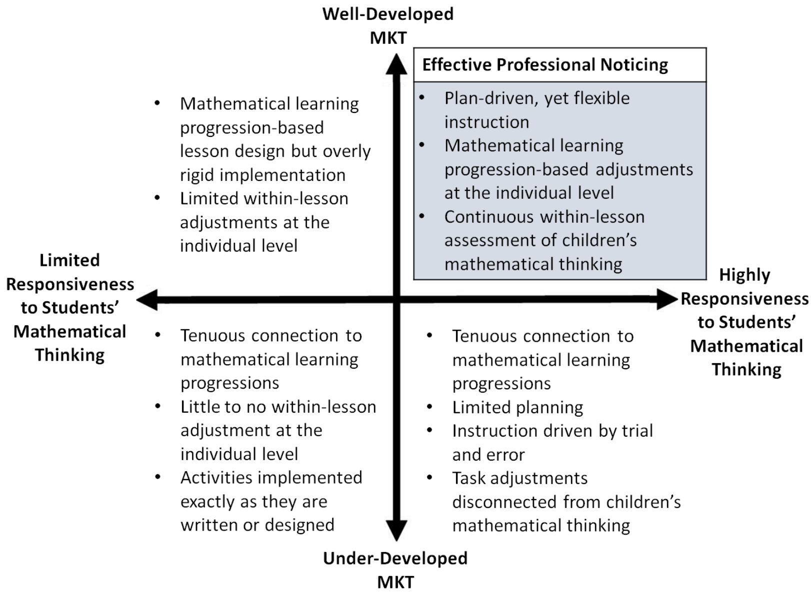

Figure 2. Enactment of professional noticing and Mathematics Knowledge for Teaching: Hypothesized Outcomes

We conjecture that effective professional noticing occurs at the intersection of developed MKT and a high level of responsiveness to the mathematical activities of students. It is important to note, however, that the figure not be interpreted in terms of dichotomies (e.g., well-developed MKT, underdeveloped MKT) as, almost assuredly, such contrasts are unrepresentative of the constructs at hand. Rather, teachers' conceptions of MKT and enactment of responsive practices most likely exist upon a continuum (or plane, in this instance) with infinitely many degrees of variation. Nonetheless, we found it useful to present the outcomes in this manner to examine the instructional implications at each quadrant of the plane as we continue to construct meaningful experiences in which to engage our PSETs. Schoenfeld (2011) sought similar affordances by positing ascending planes to describe noticing proficiency as it relates to teachers' time allocation. However, Schoenfeld's model was focused more on novice/expert transition while our model (Figure 2) aims to depict the hypothetical intersection between knowledge (MKT) and practice (noticing). On this point, we note effective professional noticing is predicated on planned instructional practices organized around one's knowledge of progressions of "children's mathematical development" (Jacobs et al., 2010, pp. 172–173); however, such practices are continually evaluated and adjusted (via the component skills of attending, interpreting, and deciding) resulting in a learning experience that is highly responsive at the individual level.

Delving deeper into responsiveness as it pertains to professional noticing, Figure 3 illustrates the manner in which PSETs' professional noticing manifests along a continuum.

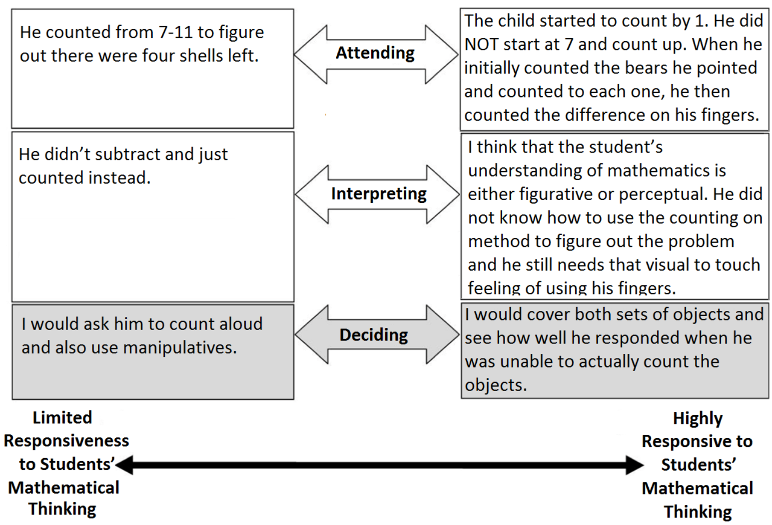

Figure 3. Examples of Nonresponsive and Responsive Professional Noticing

We conducted a study focused on the manner in which PSET's professional noticing capacities develop with respect to MKT and attitudes and beliefs towards mathematics (Schack et al., 2013; Fisher, Thomas, Jong, Schack, & Tassell, 2017). Specifically, we designed and implemented a professional learning module (embedded within methods courses) focused on attending, interpreting, and deciding in the context of early-numeracy development. Two hundred twenty-five PSETs across five public universities in a single state participated in this study. PSET performance on professional noticing component skills was measured (pre/post) via a proprietary, video-based instrument consistent with the measure used by Jacobs et al. (2010). This particular measure involved PSETs responding to open-ended attending, interpreting, and deciding prompts after watching a brief video of a child engaging in a somewhat ambiguous counting strategy while negotiating an arithmetic task (see Schack et al., 2013 for further explication of instrumentation). For Figure 3, we identified two individual PSETs who produced the highest and lowest scores for the attending, interpreting, and deciding components of a professional noticing assessment (Schack et al., 2013) to develop a post-hoc conjecture regarding the construct of responsiveness. Important for this example is the manner in which each of the PSETs' attending and interpreting potentially influences the responsiveness of the subsequent decision. For high scores on the measure, the decision component specifies the introduction of two screened collections to further examine the extent to which the child is engaging in both perceptual and figurative counting schemes (Steffe, Cobb, & von Glasersfeld, 1988; Steffe, 1992). Contrast this decision with the PSET who scored low on the measure. The PSET apparently intended to supply the child with a specific strategy to address similar tasks (e.g., "count aloud," "use manipulatives"), a strategy that seemingly reinforces counting perceived items by ones. Thus, this individual's decision is, arguably, less responsive to the observed mathematics of the child (e.g., perceptual and figurative counting schemes).

Using these cases, we observe varied levels of responsiveness in PSETs' professional noticing; however, we also conjecture that, in some cases, professional noticing performance may be considered as responsive to the learner but with limited effectiveness in terms of the ultimate instructional decision (see Figure 4).

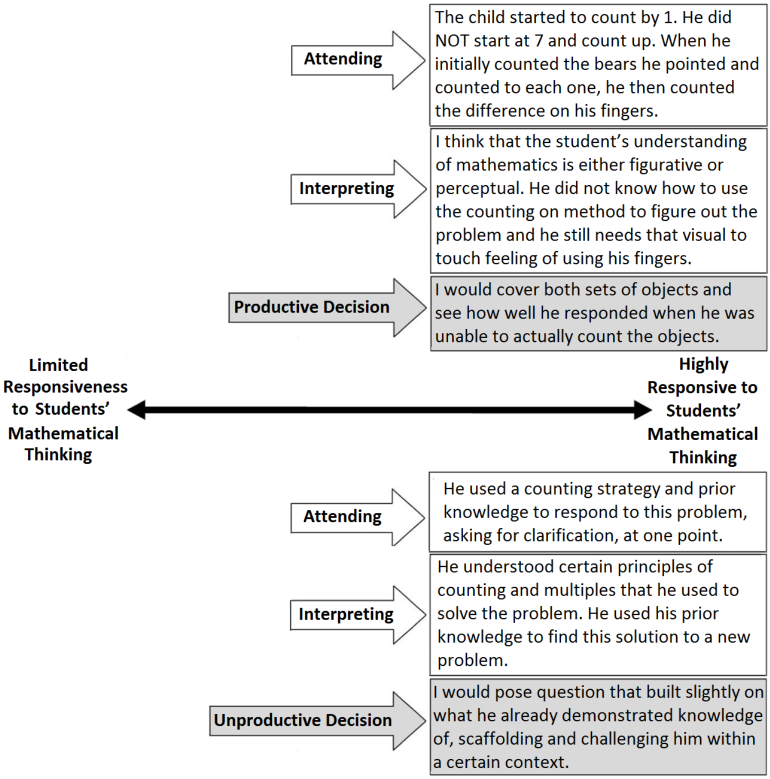

Figure 4. Examples of Productive and Unproductive Decisions

For this figure, we took the set of high-scoring responses from the previous example and identified another individual PSET whose responses also received minimum scores according to our rubric. This low-scoring individual acknowledges the child's "counting strategy," "principles of counting," and the need to pose questions that build on the child's demonstrated knowledge; however, the child's knowledge is ill-defined by the PSET in the attending and interpreting responses. Specifically, there is no connection to the child's counting scheme or conception of unit. Thus, the resulting decision, while responsive to the child, lacks necessary focus and specificity, which likely undermines its productivity.

The examination of the cases above provides an opportunity to consider specific theoretical synthesis. We contend that synthesis of professional noticing and MKT must involve some account of instructional responsiveness if one is to make any evaluative judgments regarding the quality of practical manifestation.

Empirical Study of Relationships between Professional Noticing and MKT

Preliminary investigation of relationships between professional noticing and MKT have, thus far, failed to illustrate the type of connection suggested by their commonality of construction. Mentioned earlier, we have studied relationships between PSET professional noticing capacities, MKT, and attitudes and beliefs towards mathematics in the context of a professional learning module (Fisher et al., 2017; Schack et al., 2013). In addition to the proprietary professional noticing assessment, PSET MKT was measured via the Learning Mathematics for Teaching (LMT) assessment (Hill, Schilling, & Ball, 2004; Phelps, 2011). Specifically, the Elementary–Number Concepts and Operations 2001 form was used because it offered the closest available content agreement to the early numeracy focus of the professional noticing module. Recall that results from this study indicated statistically significant positive changes in each of the component skills of professional noticing among PSETs who experienced the professional learning module.

Germane to this examination, though, are our attempts to connect professional noticing performance with MKT. Using a Spearman's correlation test, Fisher et al. (2017) determined that there were no statistically significant correlations between any of the three professional noticing components and the LMT scores on the pre-assessment. The same test was conducted using the post-test scores for the professional noticing assessment and it revealed that a statistically significant correlation was found between the PSETs' scores on the attending portion of the post-test professional noticing measure and the post-test LMT (r_s = .195, p = .003). No other significant correlations were found when comparing the professional noticing post-test scores with the LMT post-test scores.

Given the strong theoretical connections between MKT and professional noticing, such findings are, at first blush, quite surprising. However, certain mitigating factors likely influenced our inability to observe such connections. The mathematical content of the LMT, that is, K-6 number and operations, was much broader than the content taught in the module (i.e., early numeracy development). Thus, it likely would have been beneficial to employ an MKT measure that more closely aligned to the content of the noticing module. Additionally, the LMT, which was initially created for and piloted with inservice teachers, may have placed PSETs at a disadvantage because the content may be more familiar to inservice teachers who have had more opportunities to experience the mathematics scenarios in the LMT items (Hill & Ball, 2004).

Implications for Future Research

Given the abundance of theoretical frameworks within the domain of mathematics teaching and learning, identifying potential for synthesis among such frameworks is a useful first step in reducing the complexity and isolation of constructs as they relate to inquiry and practice (Prediger et al., 2008). We find synthesizing professional noticing and MKT frameworks opens broad avenues for further research. Empirically testing this theorized relationship between knowledge (MKT) and practice (professional noticing) would prove useful to solidifying the ideas presented here. Towards this end, prior examination of MKT and mathematical quality of instruction (Hill et al., 2008) might provide some guidance with respect to modes of inquiry. Although we were unable to identify empirical connections in our own research, this does not necessarily mean such connections do not exist. We identified several possible explanations for this lack of observed relationship, and our hope is that more thoughtful inquiries may uncover some links between the two similarly-constructed frameworks.

Additionally, a productive scholarly direction might focus on the relationship between responsiveness and the practice of professional noticing. Indeed, some manner of responsiveness is embedded structurally within professional noticing in the theorized relationship of the component skills of attending, interpreting, and deciding (with attending anchoring the practice firmly on students' mathematics). However, this relationship remains fairly unexplored in practice. For example, to what extent do the activities to which teachers attend and their interpretations of students' mathematical understanding need to relate to one another for the practice of noticing to be considered effective (or even viable)? Certainly, some degree of coherence is assumed across the component skills; however, such connections remain elusive. Indeed, it may be possible that this implicit notion of coherence (i.e., firmly connecting the component skills of professional noticing during enactment) is a component skill in and of itself, and that such coherence is the essence of responsiveness.

Acknowledgement

The research described in this essay was funded by the National Science Foundation via the Transforming Undergraduate STEM Education (TUES) program – awards 1043667, 1043656, and 1043831. The opinions expressed in this article do not necessarily reflect the positions or policies of the supporting agency.

References

Adams, T. L. (1998). Prospective elementary teachers' mathematics subject matter knowledge: The real number system. *Action in Teacher Education, 20,* 35–48.

Adler, J. & Davis, Z. (2006). Opening another black box: Researching mathematics for teaching in mathematics teacher education. *Journal for Research in Mathematics Education, 37,* 270–296.

Bednarz, N. & Proulx, J. (2009). Knowing and using mathematics in teaching: Conceptual and epistemological clarifications. *For the Learning of Mathematics, 29,* 11–17.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90,* 449–466.

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching (4th ed.).* New York: Macmillan.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59,* 389–407.

Ball, D. L. & Bass, H. (2009). *With an eye on the mathematical horizon: Knowing mathematics for teaching to learners'mathematical futures*. Paper prepared based on keynote address at the 43rd Jahrestagung für Didaktik der Mathematik held in Oldenburg, Germany.

Ball, D.L., & Forzani, F. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education, 60,* 497–511.

Brophy, J. and Good, T. (1986) Teacher behaviour and student achievement. In M. Wittrock (ed.), *Handbook of research on teaching.* New York: Macmillan.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L. & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.

Chick, H. L. (2009). Choice and use of examples as a window on mathematical knowledge for teaching. *For the Learning of Mathematics, 29,* 26–30.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Retrieved from http://corestandards.org/assets/CCSSI_Math%20Standards.pdf

Conference Board of Mathematical Sciences & American Mathematical Society (CBMS/AMS). (2012). The mathematical education of teachers II. *Issues in Mathematics Education,17,* 1–86.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: NY: Routledge.

Davis, B., & Renert, M. (2014). *The math teachers know: Profound understanding of emergent mathematics*. New York, NY: Routledge.

Dewey, J. (1904). The relation of theory to practice in education. In *National Society for Scientific Study of Education, The relation of theory to practice in the education of teachers* (Vol. 3, Part 1). Bloomington, IL: Public School Publishing.

Erickson, F. (2011). On noticing teacher noticing. In M.G. Sherin, V.R. Jacobs, & R.A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes*. New York: Routledge.

Fisher, M. H., Thomas, J., Jong, C., Schack, E. O., Tassell, J. (2017). Interpreting the Interrelationships of PSET Professional Noticing, Mathematics Knowledge for Teaching, and Attitudes towards Mathematics. *Mathematics Education Research Journal* [Online First Edition].

Fisher, M., Schack, E., Thomas, J., Jong, C., Eisenhardt, S., Yoder, M., & Tassell, J. (2014). Examining the Relationship Between Preservice Elementary Teachers' Attitudes Toward Mathematics and Professional Noticing Capacities. In J.J. Lo, K.R Leatham, & L.R. Van Zoest (Eds.), *Research Trends in Mathematics Teacher Education.* New York: Springer.

Foss, D. H., & Kleinsasser, C. (1996). Preservice elementary teachers' views of pedagogical and mathematical content knowledge. *Teaching and Teacher Education, 12,* 429–442.

Hanna, W. G. (2012, November). Examining what inservice elementary teachers notice when presented with instances of students mathematical thinking. *Proceedings of the 34rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kalamazoo, MI: Western Michigan University.

Harkness, S. S. & Thomas, J. (2008). Multiplication as original sin: The implications of using a case to help preservice teachers understand invented algorithms. *Journal of Mathematical Behavior, 27*, 128–137.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's Mathematics Professional Development Institutes. *Journal for Research in Mathematics Education, 35,* 330–351.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39,* 372–400.

Hill, H. C., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26,* 430–511.

Hill, H.C., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*, 371–406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 104,* 11–30.

Jacobs, V. A., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41,* 169–202.

Kersting, N. B., Givvin, K. B., Sotelo, F. L., & Stigler, J. W. (2010). Teachers' analyses of classroom video predict student learning of mathematics: Further explorations of a novel measure of teacher knowledge. *Journal of Teacher Education, 61,* 172–181.

Mason, J. (2002). *Researching your own practice: The discipline of noticing.* London: Routledge Falmer.

Mason, J. (2011). Noticing: Roots and branches. In M.G. Sherin, V.R. Jacobs, & R.A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes.* New York: Routledge.

Phelps, G. (2011). *Investigating the measurement and development of mathematical knowledge for teaching*. Paper presented at the 2011 MSP Learning Network Conference. Washington, DC.

Philipp, R.A. (2014). Research on teachers' focusing on children's thinking in learning to teach: Teacher noticing and learning trajectories. In J.J. Lo, K.R Leatham, & L.R. Van Zoest (Eds.), *Research Trends in Mathematics Teacher Education.* New York: Springer.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM, 40,* 165–178.

Quinn, R. J. (1997). Effects of mathematics methods courses on the mathematical attitudes and content knowledge of preservice teachers. *The Journal of Educational Research, 91,* 108–113.

Russell, J., Anderson, R, Goodman, J. T., & Lovin, L. (2009, October). *Assessing mathematical knowledge for teaching: An examination of the Learning Mathematics for Teaching Instrument*. Paper presented at the annual meeting of the Northeastern Educational Research Association, Rocky Hill, CT.

Schack, E.O., Fisher, M.H., & Wilhelm, J. (2017). *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks.* New York: Springer.

Schack, E., Fisher, M., Thomas, J., Eisenhardt, S., Tassell, J., & Yoder, M., (2013). Preservice teachers professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education, 16,* 379–397.

Schifter, D., Bastable, V., & Russell, S. J. (2000). *Developing mathematical ideas.* Columbus, OH: Pearson Education.

Schoenfeld, A. (2011). Noticing matters. A lot. Now what? In M.G. Sherin, V.R. Jacobs, & R.A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes.* New York: Routledge.

Seago, N., Mumme, J., & Branca, N. (2004). *Learning and teaching linear functions: Video cases for mathematics professional development.* Portsmouth, NH: Heinemann.

Shechtman, N., Roschelle, J., Haertel, G. & Knudsen, J. (2010). Investigating links from teacher knowledge, to classroom practice, to student learning in the instructional system of the middle-school mathematics classroom. *Cognition and Instruction, 28,* 317–359.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.) (2011). *Mathematics teacher noticing: Seeing through teachers' eyes.* New York: Routledge.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60,* 20–37.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4–14.

Steffe, L. (1992). Learning stages in the construction of the number sequence. In J. Bideaud, C. Meljac, & J. Fischer (Eds.), *Pathways to number: Children's developing numerical abilities* (pp. 83–88). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Stein, M. K., Engle, R. A., Smith, P. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10,* 313–340.

Stoddart, T., Connell, M. L., & Stofflett, R. T. (1993). Reconstructing elementary teacher candidates' understanding of mathematics and science content. *Teaching and Teacher Education, 9,* 229–241.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Towards a theory of teaching. *Educational Researcher, 41,* 147–156.

Thomas, J. & Tabor, P.D. (2012). Developing Quantitative Mental Imagery. *Teaching Children Mathematics, 19,* 174–183.

Towers, J. & Martin, L. (2009). The emergence of a 'better' idea: Preservice teachers' growing understanding of mathematics for teaching. *For the Learning of Mathematics, 29,* 44–48.

Tsamir, P. (2005). Enhancing prospective teachers' knowledge of learners intuitive conceptions: The case of Same A – Same B. *Journal of Mathematics Teacher Education, 8,* 469–497.

Waxman, H.C., Wang, M.C., Anderson, K.A., & Walberg, H.J. (1985). Synthesis of research on the effects of adaptive education. *Educational Leadership, 43,* 27–29.

Westwood, P. (1996). Effective teaching. *Australian Journal of Teacher Education, 21,* 66–84.

Wickstrom, M. H., Baek, J., Barrett, J. E., Cullen, C. J., & Tobias, J. M. (2012). Teacher noticing of children's understanding of linear measurement. *Proceedings of the 34rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kalamazoo, MI: Western Michigan University.