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Taking the Guesswork out of Computational Estimation

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Computational estimation is an important skill necessary for students' mathematical development. Students who can estimate well for computations rely on an understanding of many mathematical topics, including a strong number sense, which facilitates understanding the mathematical operations and contextual evidence within a problem. In turn, good estimation skills improve students' ability to do exact computations and help them determine the reasonableness of their solutions. There exists research regarding estimation, its significance, and some of the possible estimation strategies (National Council of Teachers of Mathematics, 2000; Sowder & Wheeler, 1989; Van de Walle, Karp, & Bay-Williams, 2010). While this research exists, it does not adequately address the particular ways students understand and perform estimations. In this study, we asked middle school students to answer both written and oral problems, which suggested the use of estimation. Our findings contribute to a more comprehensive definition of computational estimation and its strategies by presenting student examples. These student examples serve as a model for identifying the significance of computational estimation and its link to other conceptual mathematical knowledge.

Background

Estimation is important for helping students to develop number sense and the ability to evaluate the reasonableness of their solutions. According to the National Council of Teachers of Mathematics (NCTM), many students do not develop a good

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understanding of estimation. The NCTM (2000) found that "when asked to estimate, only 24 percent of thirteen-year-old students in a national assessment said the answer was close to 2" (p. 35). Students that are able to estimate successfully show an understanding of the value of numbers and of the operations used. Furthermore, the NCTM states that students should be able to "compute fluently and make reasonable estimates [as well as] develop and use strategies to estimate the results of rational-number computations and judge the reasonableness of the results" (p. 393). As a result of this statement of significance set forth by the NCTM, several other researchers have explored strategies involved in computational estimation (Sowder & Wheeler, 1989; Van de Walle, Karp, & Bay-Williams., 2010). In this article, we elaborate upon the definitions of computational estimation strategies by providing middle school students' work examples that utilize particular estimation strategies. Through these samples, we demonstrate the application of various computational estimation strategies previously defined in literature. In addition, we identify computational estimation strategies not previously defined and discuss the link each strategy has to conceptual mathematical knowledge. Developing a better understanding of students' computational estimation strategies and the connections these strategies have to various mathematical concepts is critical for improving the teaching of computational estimation.

Estimation Literature Review

Based upon literature and previous research on estimation, we can begin to define what estimation is and is not. It is more than a guess and more than simply rounding, yet it is not an exact answer (Sowder & Wheeler, 1989). Van de Walle, et al. (2010) further defined estimation by saying, "estimation refers to a number that is a suitable approximation for an exact number given a particular context" (p. 241). We can refine the definition of estimation more specifically by considering *computational estimation*. Computational estimation occurs when either the determination of the problem is too complex by exact methods or the problem does not require an exact answer given the context of the situation (Van de Walle et al., 2010).

This definition illuminates one important goal for student estimation. That is, that students should be able to make sense of what a problem is asking based on the context of the question. The goal of utilizing the context of a question is not only relevant to estimation, but also to mathematics more generally and is an example of the type of conceptual knowledge required for competence in estimation (Van de Walle et al., 2010).

We feel it is important to take an in depth look at the conceptual knowledge of estimation needed to successfully estimate computations, as this directly relate to the overall goals of teaching estimation. Before estimating, there are three concepts that students should understand in order to estimate appropriately (Sowder & Wheeler, 1989). Students should recognize approximate numbers that they can estimate in multiple ways and receive multiple answers, and when estimation is an appropriate tool for problem solving based on the context of the problem (Lefevre, Greeenham, & Waheed, 1993; Sowder & Wheeler, 1989).

Once students have constructed understandings of the three basic concepts of estimation, students make use of a few other mathematical concepts when performing computational estimation. These concepts are as follows: knowledge of arithmetic facts, fluency in mental computations, an understanding of the base-10 number system, an understanding of place value, and an understanding of the relative sizes of numbers (Lefevre et al., 1993; Rubenstein, 1985; Sowder &Wheeler, 1989). Therefore, another important goal associated with computational estimation in particular, is helping students develop the ability to utilize these concepts to the benefit of their estimates.

Now that we have explored various definitions of estimation, the mathematical skills estimation involves, and the overall goals of computational estimation, we can look more closely into specific computational estimation strategies. Sowder and Wheeler (1989), as well as Van de Walle et al. (2010), suggested the following possible methods: rounding, benchmarks, compatible numbers, front-end, clustering (averaging), and adjusting (compensation). Figure 1 displays the definition of each strategy and each of these methods will be expanded upon later through examples of student work.

<u>Rounding</u>—moving a number to the closest whole number to make computation "easier." <u>Benchmarks</u>—using knowledge of the base-ten system when rounding to make a problem simpler. <u>Compatible Numbers</u>—manipulating numbers, in division, so that both numbers in a given problem are evenly divide into one another. In addition and subtraction the compatibles method can be used when a student recognizes two numbers that can add up or subtract to make a benchmark. <u>Front-End Method</u>—solving with the front-digits first; working backwards on a problem. <u>Clustering (Averaging)</u>—looking for an estimate of an average in a data set by looking for a number that all other numbers seem to surround. <u>Adjusting/Compensation</u>—when one factor is changed in a problem to make it easier to solve and then the other factor in the problem is manipulated to adjust for that previous change.

Figure 1. Computational estimation strategy definitions adapted from Sowder and Wheeler (1989) and Van de Walle et al. (2010).

Methods

In order to identify students' use of various computational estimation strategies, we worked with 26 students in sixth, seventh, and eighth grades at a private school located in a rural community in the southeastern United States. The interviews were approximately 30 minutes long and lasted over two weeks during the spring of 2012.

We split our time with the students into two components: one in which the students solved written problems and another in which the students talked through their solutions to math problems (see Table 1 for the specific questions used in each format). We first gave the written format questions to all of the students at each grade level. In addition to informally evaluating whether a student understood when an estimate was appropriate, the written section questions allowed us to examine students' particular strategies for estimating.

In contrast, we designed the verbal format questions to better understand student reasoning. We both read the questions aloud and presented them on paper for the student to follow along visually. The student then described his or her mathematical thinking and showed the work on paper. We documented student responses via interview notes. We chose students to participate in the verbal section if they utilized a previously defined estimation strategy or a potentially invented strategy that required further explanation. Of the 26 students that took part in the study, we chose 13 for the verbal interview. The verbal questions and subsequent conversations focused on prompting the students to explain their reasoning by asking them to assume the role of teacher. We also looked for multiple strategy recognition, a key skill in estimating which involves the recognition that more than one strategy can be used in solving a math problem (Sowder & Wheeler, 1989).

We based the question formation on the definitions and previous research described in the literature review. We questioned the students using situations that implied an approximate estimate as an answer, but did not explicitly ask the students to estimate. The reason for this was to assess whether the students could understand contextual clues indicating that an estimate was appropriate, since this was a goal found in the estimation definitions. We wrote the items so that particular estimation strategies would fit well with specific questions. For example, in order to offer the opportunity for students to use the averaging strategy, we created Written Format Question 2 (see Table 1), which asks for an average out of a set of data. The bold text under each question in Table 1, which we did not share with the students, indicates the strategies that each question highlighted.

Results: Strategies Used by Middle Grades Students

The students' answers from the verbal component of the interviews were helpful in elaborating the definitions for the different estimation techniques. In addition, some of the student responses showed that those students had formed links

Table 1.

Computational Estimation Instruments

Written Format Estimation Questions

1. There are 374 seats at a local musical theater and each row has about 6 seats. Approximately how many rows are there? How do you know? **(Compatible numbers)**

2. Brittany and Joe have been measuring the height of their sunflower once a week to see its rate of growth. They want to see on average how much their plant grows per week, but they cannot find their calculator. How would they quickly figure out about how much the rate of average growth is? How do you know that your answer is the approximate average? **(Clustering/Averaging)**

Week	Growth
1	$2\frac{7}{8}$ inches
2	$3\frac{1}{2}$inches
3	$2\frac{1}{2}$ inches
4	3 inches
5	$2\frac{3}{8}$ inches
6	$3\frac{3}{8}$ inches
7	$2\frac{1}{4}$ inches

3. Jordan is running a fundraiser and wants to quickly tell everyone about how much all of the groups raised for charity. About how much would you say all of the groups earned? How did you get that total? **(Front-End, Rounding, Compatible numbers)**

Group	Total Money Raised
1	2,672
2	573
3	1,087
4	796
5	998

Verbal Format Estimation Questions

1. Your younger sibling is having trouble with some math homework one night and your mother asks you to help because you are really great at math. Your younger sibling's first math problem says the following:

   Suzanne is at the computer store and sees a computer that is normally $325.72. It is now half price because of a weekend sale. About how much is the computer, now that it is on sale?

   Your younger sibling is confused because he/she says that the class has never worked with such large and complicated numbers before. He/she also says that the teacher told them not to use a calculator! What would you advise your sibling to do?

   (Benchmarks, Rounding, Compatible numbers)

2. At the beginning of math class one day, your teacher places a math problem on the board. You are placed into groups of four to discuss the problem and come up with a group answer. The question is:

   Brian owns a catering business and was just hired for a very large birthday party. The people throwing the party order 37 party platters. The people ordering the platters want Brian to quickly tell them about how much the total is for their order. What should Brian tell them if he knows that 1 platter costs $11.56?

   a. What would your answer be? Why? How did you get that answer? **(Benchmarks, Rounding, Compatible numbers)**

   b. A girl in your group says that Brian should say that it will cost them about $370. Another girl in your group says that Brian should find a calculator. The third person in your group says that the answer is $380. What would your answer be? Is anyone in the group wrong? Is there an answer that you feel is best?**(Multiple strategy recognition)**

between computational estimation and other types of conceptual math knowledge. The responses also demonstrated what students think about estimating and how students can link certain techniques to enable greater estimation ability. Figure 2, shows the overall results of the study and the following sections include examples of students using each of the estimation strategies.

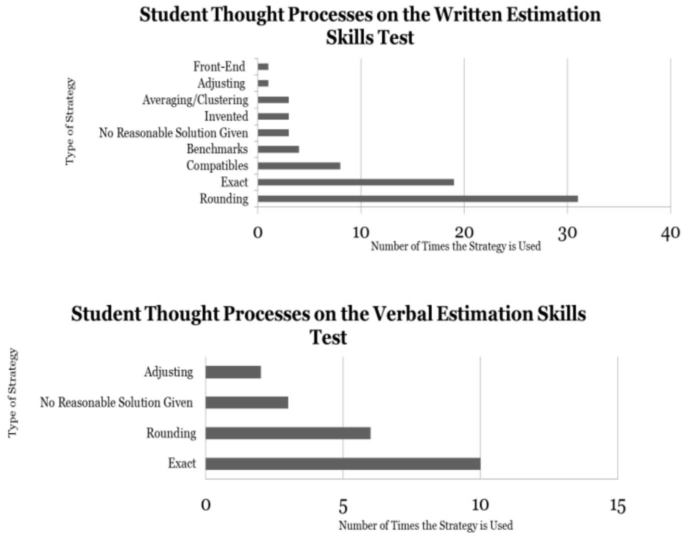

Figure 2. Overall student strategy usage.

Rounding and Benchmarks

In terms of estimation strategies, it is important to first look at the two most commonly used and well-known strategies of the students within the study—rounding and benchmarks. In fact, we found that although we wrote the questions to encourage the use of a variety of estimation methods, rounding was used most often, followed by using an exact method (see Figure 2 for complete frequencies of strategy usage). A n eighth grade student's response to Verbal Format Question 1(see Table 1) demonstrates both rounding and benchmarking. This student responded, "I would round [325].72 up to 326, but since my sibling hasn't worked with large numbers, I would round to 300 so that their answer would be $150". Here the student shows multiple strategy recognition. She demonstrated traditional rounding in the first response but also recognized the real world application of the question and the need to make it simpler for her younger sibling. In performing this second rounding, the student used benchmarking and her knowledge of the base-10 number system to simplify the question further.

Compatible Numbers

A seventh grade student utilized the compatible number strategy for division (see Figure 3) in response to Written Format Question 1 (see Table 1). This question asked for the approximate number of rows with 374 seats arranged in rows of 6. Here the student looked for numbers close to 374 and 6 that could evenly divide. He additionally used benchmarks and divided 370 by 10 rather than 374 by 6. While the student's response does show a significant margin of error, his method shows conceptual knowledge of the base-10 number system and arithmetic facts. We asked students regarding the reasonableness of their answer and whether they could find another way to solve the problem. This student responded that the answer was reasonable and that although other methods of solving were possible, this was the only way he could think of at that time.

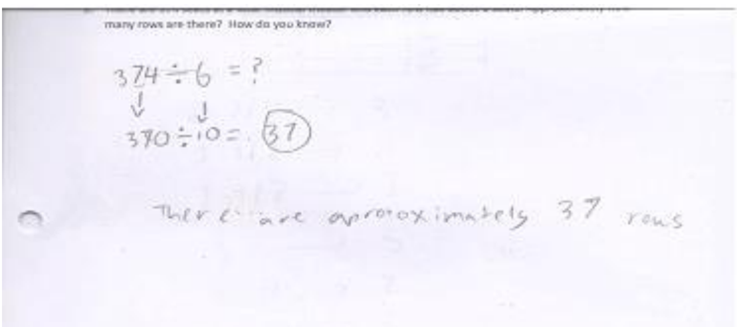

Figure 3. Student work using compatible number strategy.

Front-end Method

A sixth grade student demonstrated the front-end method when answering Written Format Question 3 (see Table 1), which asks for an estimate of the total money raised for charity. The money raised by 5 different groups were as follows: $2672, $573, $1087, $796, and $998. For this question, the student demonstrated the front-end method by first adding

2,000 and 1,000 to get 3,000. Then by working backwards (i.e. left to right) , the student recognizing that 998 and 796 are both close to a thousand which would then yield a running total of 5,000. Finally, the student saw 573 and looked at the 600 in the hundreds place of 2,672 and recognized that adding the hundreds place for these numbers is about 1000. This process resulted in the student's final estimate of $6,000 for the total.

Clustering (Averaging)

A sixth grade student explained this method best in response to Written Format Question 2 (see Table 1). This question contains a list of the amounts of plant growth that the students recorded at the end of each week. The student looked at the list of numbers, recognized that all of the numbers seemed to cluster around three, and therefore estimated an average of three. This may seem to be a simple method of estimation; however, it requires the conceptual understanding that the average is the "middle" of the data. Many students struggled with this understanding of average and added the numbers to find the total growth of the plant instead of the average growth. This misconception demonstrates the importance of a conceptual understanding of averages in order to estimate using this method.

Beyond the examples of computational estimation strategies already described, adjusting (compensation) is an additional step that a student can take with any estimation strategy. A seventh grade student used this method in finding his answer to Verbal Format Question 1. This question asked for an estimate for half price of a $325.72 computer in the context of explaining the estimate to a younger sibling. The student responded

> I would probably say round to $300 if it doesn't have to be exact. So that would be $150 and then take a little higher than $150. If they were really young then, round to $400. If they are closer to my age then they can round to $350.

In this example, the student recognized that his answer was going to be low because he initially rounded down to $300. Therefore, he adjusted the estimate in saying it would be a little larger than $150. The student additionally recognized that there were several different ways to adjust the initial cost based on the context of the situation.

Another example of adjusting was shown by this same student in Verbal Format Question 2a (see Table 1). This particular question looked for an estimated total cost for 37 party platters that are $11.56 each. He answered, “If it needs to be quick then I would round up to 40 and down to $11. It’s $440.” When asked why he rounded one down and another up the student replied, “If I rounded them both up or down it would change the price more. In the real world I would round down to make it seem cheaper.” There are several important reasons to pause and analyze this response. First, the student did not adjust the resulting answer, but rather he did so during the computational process. When utilizing the rounding strategy, one might round 11.56 up to 12 rather than down to 11 because 11.56 is closer to 12. In contrast, one who utilizes the compensating strategy recognizes that rounding both answers up may alter the estimation and bring it farther from the actual answer. Finally, he showed further signs of advanced estimation skill through applying the problem to a real world situation and using the context to justify his estimate.

These two sample answers utilized adjusting and compensation, showed an understanding of numeric relationships, and allowed the student to reach even closer estimates. An advanced estimator can manipulate numbers according to the given situation in this fashion. This is because, as Sowder and Wheeler (1989) stated, an estimator has knowledge of the arithmetic facts necessary to solve the problem, the ability to compute mentally, and the ability to make size comparisons. Since an advanced estimator sees number relationships, he or she can adjust and compensate an answer in multiple ways, allowing for multiple strategy recognition. In summary, an adjustor/compensator solves and adjusts according to the context of a question.

A reliance upon an Exact Method

While working with the middle school students, we noticed a tendency for many students to use exact computation instead of an estimation strategy. They did this even though the questions strongly suggested that estimation was appropriate. Unfortunately, in many cases, students who chose exact methods struggled to solve the problem or came up with an answer that was not reasonable. The following are two examples of students who struggled with estimation concepts. The first is a response to Verbal Format Question 1 (see Table 1), which involves finding half the price of a $325.72 computer. When asked this question, a sixth grade student answered as follows: “You would multiply $325.72 by…no…you would reduce the number by 2 … wait … I’m sorry. You would divide by 5 because 5 goes into 35…wait by 8 because there are 32” (see Figure 4 for this student’s final written work). It is clear that the student struggled with a conceptual understanding of half as well as recognizing that the question implied estimation.

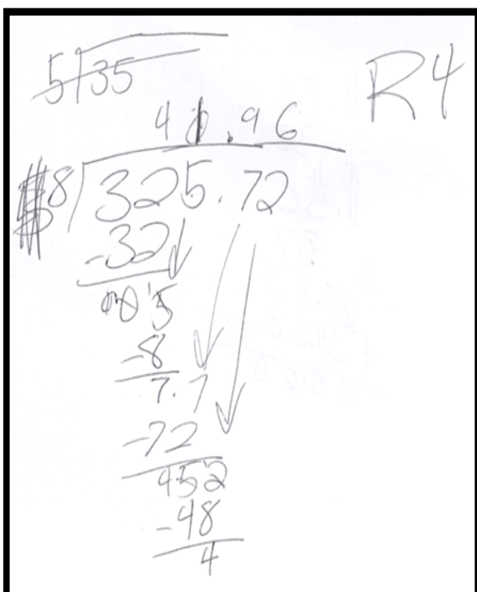

Figure 4. Student work solving for half of $325.72

Another sixth grade student used an exact method rather than estimating when asked to estimate half of $325.72. This student found the exact answer despite the implication to estimate and responded, “Split .72 in half giving .36; split 25 in half giving 12.50; split 300 in half giving 150. So then you have $162.86. I am always exact out of habit." Although this was a knowledgeable method for finding the exact solution, the student did not estimate, acknowledging the use of an exact method.

Discussion

The purpose of the estimation interviews with the middle school students was to demonstrate student applications of previously defined estimation strategies. Through observing students' use of these strategies and presenting examples of their work, we have been able to look into how students' use of computational estimation strategies is linked to other mathematical concepts. While the interview questions highlighted specific strategies, a variety of estimation strategies was acceptable and utilized for each question. However, when considering the overall frequencies of strategy usage (see Figure 2), there was a strong reliance on rounding and using exact methods. Although rounding is an acceptable method, it is important that students are flexible in their use of strategies, which might better fit the given context. Further, it is important to investigate the students' strong reliance upon the exact method. Paul (2011) suggests that this reliance upon exact calculations comes from societal reliance on exact numbers and solutions.

In addition, we had anticipated that competent estimators would invent their own estimation strategies. Invented strategies do not match any of the estimation strategies previously listed, but instead, the student comes up with his or her own creative and unique approach. We believe it is important to recognize that the strategies used in this article are not the only ways that estimation problems can be solved. Although we did not encounter any invented methods, we anticipate there could be several possible strategies that students could invent for solving these problems. We believe that as students develop greater fluency with computations estimation they will become more flexible in their estimation and begin to move beyond the previously defined strategies.

Concluding Thoughts

Through analyzing the literature of others regarding estimation and including student work samples from our interviews with middle school students, we contributed a more comprehensive definition of the strategies of estimation. By analyzing student examples, the definitions became more robust and detailed while also demonstrating the variety of approaches students used. Based on the reasonableness of different student responses, there is evidence to suggest that good student estimators use multiple strategies and can even combine methods. One example demonstrating this concept is the student who utilized adjusting in combination with the use of rounding and benchmarks. By looking at student example work such as that of the adjustor, we were able to attain knowledge related to the link that computational estimation has to other conceptual mathematical areas.

It is important to integrate estimation and the exploration of particular estimation strategies into mathematics instruction and we feel that a productive next step would be to explore how to improve instruction of computational estimation in schools and to investigate innovative ways of teaching estimation. For example, offering instruction with real world applications of estimations. Estimation and number sense go hand-in-hand. If a student does not have "quantitative intuition," they cannot construct a reasonable estimate and vice-versa (Pike & Forrester, 1996, p. 43). If we recognize this correlation and work with students' estimation strategies, in combination with number sense concepts, we can better facilitate students' development of competence in estimation.

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