The Mathematics Educator
2005, Vol. 15, No. 2, 17-24

How Can Geometry Students Understand What It Means to *Define* in Mathematics?

Patricio Herbst
Gloriana Gonzalez
Michele Macke

This article discusses how a teacher can prepare the terrain for students to understand what it means to define a figure. Drawing on writings from mathematicians and mathematics educators on the role of definitions in mathematics, the authors argue that students develop a greater appreciation for the conciseness of a mathematical definition if they are involved in activities of generating figures that meet stipulated properties. The authors illustrate that argument with episodes from students' play of a game called Guess My Quadrilateral! in two high school geometry classes.

Mathematical definitions can be described logically as the statement of the necessary and sufficient conditions that an object must meet to be labeled by a certain word or expression. Thus the expression "circle of center O and radius r" can be defined as the set of all points in a plane that are at a distance r from a given point O. This definition means: (1) that it suffices for a point to be at a distance r from O to be a point on the circle (or, no other condition is needed), and (2) that if a point is on the circle, its distance from O is necessarily equal to r (or, this condition cannot fail). In his essay on "Mathematical Definitions and Education," however, Henri Poincaré (1914/2001) noted that compliance with logical stipulations is not enough:

> A definition is stated as a convention, but the majority of minds will revolt if you try to impose it upon them as an *arbitrary* convention… Why should these elements be assembled in this manner? … What need does it fill? … If the statement is sufficiently exact to please the logician, the [answer to those other questions is what] will satisfy the intuitionist. But we can do better still. Whenever it is possible, the justification will precede the statement and prepare it. (p. 452)

Among the things that an intuitionist would appreciate about a definition, Poincaré noted, was the sense in which the object being defined was different that other neighboring objects:

> The definition will not be understood until you have shown not only the object defined, but the neighboring objects from which it has to be distinguished, until you have made it possible to grasp the difference, and have added explicitly your reason for saying this or that in stating the definition. (Poincaré, 1914/2001, p. 452)

Those comments are especially appropriate with regard to students' learning of definitions for geometric objects in the high school geometry class, suggesting that students will grasp what the definitions of particular geometric objects mean only if they also learn what it means to define a mathematical concept.

Pat Herbst is an associate professor of mathematics education at the University of Michigan and former high school teacher in Argentina.
Gloriana Gonzalez is a doctoral student in mathematics education at the University of Michigan and former high school geometry teacher in Puerto Rico and Massachusetts.
Michele Macke is a high school mathematics teacher in Michigan.

Definitions of Geometric Figures and Students' Prior Experiences

Two elements of students' prior knowledge seem to make this learning difficult. On the one hand, students come to us with some idea of what it means to **define a word**. These ideas are derived from their experiences in the highly verbal adult world. Students encounter many new words in their natural language as they read texts (take, for example, words like *pollution* or *democracy*). They wonder what those words mean and often relate to those words through the explanations of more competent speakers. In briefing them on what a word like *pollution* means, someone (or the dictionary, eventually) might try to spell out as much as can be said about the new word to foster understanding and proper usage, giving general statements, or alternative general statements, as well as particular examples of correct usage. If defining a word means spelling out what it means and enabling the audience to use it competently, it seems as though there is no reason to prefer succinct definitions. In that sense, mathematical definitions differ from the definitions of ordinary words.

On the other hand, students also come to us with a wealth of knowledge about geometric figures that they have been building since their toddler years and on through elementary and middle school. They have been naming figures by pointing to objects and using those words in geometric activity. They have a sense of familiarity with geometric figures that may conspire against our desire to develop in students the sense that definitions are needed. Indeed, if you started your first day of class asking your students to draw a circle, you would not very likely hear a student asking you "what do you mean by circle?" Words like circle, square, rectangle, or kite are familiar to them—to the point that students can use many of them without really questioning whether they know what those words mean. Therefore, as teachers of high school geometry, we could hardly claim that when we teach students the definitions of many of the geometric shapes, we are answering any question they might have as to what those words *mean*.

The instructional problem that we want to discuss derives from the foregoing discussion. Within the world of familiar objects and their names, the world of geometric figures, how can we create in students the sense that definitions are needed? Furthermore, how can students develop an appreciation for mathematical definitions—that is, statements that neither simply name nor describe beyond doubt but rather provide necessary and sufficient conditions for assigning a name to one kind of object and not to many others that could be similar to it in some respect?

Making Definitions: Descriptive and Generative Activities

The high school geometry course has usually been predicated as an opportunity to expose students to an example of a mathematical system, whereby they ought to learn to relate to figures not by what their names implicitly evoke but by what their definitions explicitly require. Edwin Moise expressed the following remark with which we agree:

> The intuitive immediacy of geometric concepts has another important pedagogic consequence. One of the vital processes in creative mathematical work is the transition from an intuitive idea to an exact definition. (Moise, 1975, p. 476)

What kind of instructional activities can summon what children know about geometric figures and their names and at the same time give the teacher leverage for promoting an overhaul of what students understand by *define*? How can one satisfy Poincaré's expectations, justifying the need for each of the stipulations involved in a definition and clarifying the differences between the object defined and its neighboring objects? Moise suggested that activities of **describing the properties of a figure given through pictures** could summon students' intuitions of the figure being defined:

> Nearly every geometric definition can be—and commonly is—preceded by a picture that conveys an intuitive idea. The definition can then be checked against the pictures, with a view to finding out whether the definition really describes the idea that it is supposed to describe. (p. 476)

One can probably play out a scenario in which students could take an active role in the description: The teacher shows several examples of rectangles — some large, some small, some with consecutive sides of very different length, some of very similar length. Then the teacher asks students to say what is common to all of the rectangles. If the students only mention properties that are common to parallelograms, the teacher may pull a parallelogram that is not a rectangle and ask students to comment on differences between the new shape and the ones on the board. This may strike some as a commonplace occurrence, for example, the social studies teacher who develops concepts of abstract ideas (e.g., *democracy*) by exposing his or her students to different examples and asking them to describe what they have in common might recognize that practice as familiar. And yet it may also strike the mathematics teacher as an unintelligent strategy in regard to how to make students aware of the difference between a description and a definition. Definitions are not descriptions—at stake in a definition is not just clarity on the use of a word but also succinctness of formulation. How can students develop an appreciation of the power of statements that balance clarity with succinctness?

We contend that to help students develop appreciation of the various aspects that Poincaré saw involved in the work of giving a definition, engaging them in the activity of describing figures is not enough. They also need to be engaged in activities of **generating figures by the properties they should have**. By *generating* (or prescribing) a figure we mean stipulating the conditions that a hypothetical figure would have to satisfy, and then finding whether a figure exists that satisfies such conditions. In this article we want to elaborate on that point, showing an example of an activity of generating a figure that we found useful in the process of engaging students in defining. We show an example of how we chose to organize this kind of activity as we introduced students to the definitions of some of the special quadrilaterals.

The Quadrilaterals Unit and the "Guess My Quadrilateral!" Game

The teaching idea that we present was part of a unit that we designed and implemented in two accelerated geometry classes populated by a total of 53 students, mostly 9th graders, in a very large and diverse public high school. In planning a unit on quadrilaterals we posed to ourselves the instructional problem of how to create a context for students to think of the minimal conditions that a shape must satisfy in order to be one of the special quadrilaterals. We acknowledged the likelihood that students would understand what the special quadrilaterals were and set out to use (rather than ignore) that prior knowledge. We were actually able to confirm this prior knowledge using a questionnaire that we gave to the two classes before the unit (see Appendix A).

We observed that students knew quite a bit about squares, rectangles, and rhombi. For example, half of the students knew that diagonals of a rhombus are perpendicular and two thirds of the students knew that such was not the case for a rectangle. They also did not know some things—e.g., two thirds of the students did not think that diagonals of a rectangle are congruent—or had misconceptions—e.g., one third of the students asserted that diagonals of a rhombus are congruent. In planning the unit, we also expected that merely asking them to define those figures would not provide enough support for them to think of the differences between defining a mathematical object and defining an unknown word. The initial questionnaire actually confirmed the notion that students naturally associate the word *definition* with either 'say as much as you know' or 'point to an object that bears this name.' For a question that asked them to define a rectangle, we found that 88% of the students were evenly split among those that would provide too much information (e.g., a quadrilateral with two sets of parallel sides and four right angles) and those that would provide insufficient information (e.g., a quadrilateral with two sets of parallel sides), while only 12% provided definitions that were necessary and sufficient. It seemed as though students did have intuitions of what the figures are but if they were to appreciate mathematical definitions that provide necessary and sufficient conditions they would need help understanding in what sense one such statement is better than another one which says as much as one knows or one that names and points.

The lesson that we share here was part of a special unit on quadrilaterals that we designed and implemented over a period of three weeks. Rather than introducing one special quadrilateral at a time, only the parallelogram was studied on the first day of the unit. A few days after having created a long list of possible properties a quadrilateral could have, students were introduced to defining special quadrilaterals by way of a game that we called Guess My Quadrilateral! This game, we contend, provided an effective context for students to understand why a statement that provided necessary and sufficient conditions might be preferable to one that spells out as much as one knows. The game also created a context for students to deal with the whole neighborhood of special quadrilaterals and thus concentrate on what about each of them made it different from its neighbors. As a pedagogical strategy, the Guess My Quadrilateral! game is an example of an activity in which students are engaged in generating a figure. The play of the game itself and the discussions about playing that ensued eventually gave rise to the definition of each of the special quadrilaterals; this took most of two days. In the following paragraphs we describe what we planned and show examples of how students worked on it.

The game consisted of having each group of students determine the name of a shape they could not see. A generic representation of the figure had been drawn on a card that the teacher kept out of the students' sight. Students could gather information about the unknown figure by asking the teacher questions that admitted only "yes" or "no" for answers. Groups played in parallel, each group asking the teacher questions about their shape; but groups competed against each other in being able to guess their own shapes while asking the minimal number of questions. Groups accrued 1 point per question they asked; and at anytime they decided they knew the shape, they could make a guess. Their questions could ask for any property from the list of possible properties they had generated some days before. In this way, students could ask questions like "does it have a right angle?" but not questions like "what are the measures of its angles?" (since the latter question could not be answered with a "yes" or "no"). Also, whereas it was okay for them to ask whether the unknown figure was a specific quadrilateral (e.g., "is it a rhombus?"), if the answer to such a question happened to be negative, the group would accrue 999 points—thus they were discouraged from blindly guessing and encouraged to make sure their eventual "guess" would just be a check against a near certainty. After each group had the chance to play three times, the total scores for each group would be compared; and whichever group had made the least total number of points would win the game.

As far as managing the play of the game was concerned, to make sure students understood how to play, we had one group do a trial run of the game in front of the whole class. After that demonstration, we had students in their groups write their questions on easel pad sheets; and when they were ready to ask their first question, the teacher would visit the group, answer the question, and they would write that answer next to the question. At the end, after they had made a correct or incorrect guess about the identity of the figure, we would stick the picture of the previously unknown figure to the sheet. After all groups had guessed the identity of the figure, we would stick the sheets to the blackboard, enabling the students to see the ways in which people had played the game as a segue into definitions for the shapes.

The Game: An Example of Engaging Students in Generating a Figure

What is it about this game that may help create a context for discussion about definitions? The game is meant to engage students with figures in a different way than usual. The game makes students generate a figure rather than describe it, which we contend provides a meaningful context for students to understand what it means to define in mathematics. When students come to high school geometry, their familiarity with figures has been shaped by tasks that rely to a large extent on *seeing* the figures as they talk about them. In contrast, generating a figure relies on talking about the attributes of figures that cannot yet be seen; figures must be imagined on the basis of what the prescribed properties allow. The Guess My Quadrilateral! game helps convey the idea that the properties that are true of a figure are the ones that make a figure what it is. Furthermore the game provides a way to gauge the extent to which more information is needed in order to know what a figure is. By making students accrue points (which negatively affect their chance to win the game), the game allows students to realize that it might be a good idea to ask questions that add really important information. The existence of an actual card with the figure drawn on it helps ensure that whoever (the teacher or another student) responds to questions would do so fairly—in our case it helped students see the teacher as less of an oracle and more like a device of the game[1].

Learning to Define by Playing the Game

In playing Guess my Quadrilateral!, students confronted the need to stipulate conditions that would actually constrain what the unknown figure could be, differentiating it from all the other figures that it could possibly be without extra stipulations. As we saw them doing this, we recognized they were doing what Poincaré had identified as central in the work of defining—distinguishing an object from its neighboring objects. This thinking was visible as they discussed in their groups what kind of questions they should ask.

For example, a group composed of four students—Alana, Madeleine. Pavan, and Tobey[2]—started debating whether they should ask a question that would determine if the unknown figure was a square; thus, they started going through the properties that would be true in the case of the square. First, Tobey suggested asking whether the figure had two sets of parallel sides, then Pavan suggested that they should ask whether all angles were congruent because if the response were negative they would be able to eliminate the square as well as the rectangle. Alana reaffirmed that suggestion by indicating that they "need to ask a question that will eliminate the most" shapes. They implemented this idea by asking whether all angles were congruent and used the response to decide where to go next. Thus, after a negative response they asked whether the figure had two pairs of opposite congruent angles; and after a positive response, they asked whether the sides were all congruent. Of the 13 groups that played the game in the two classes, five actually arrived at a rather complex decision tree for what question to ask at any given time, considering the responses to previous questions and converging to decisions on the unknown quadrilateral. (Figure 1, which was drawn to display the work turned in by the group of Heidi, Jessica, Mitchell, and Neil, gives an example of such a decision tree).

Students came to appreciate the cost and value of succinctness as they faced responses to complex questions. Groups did not always choose to ask questions that eliminated half of the available alternatives but rather questions that were engineered to identify one shape. Heidi, Jessica, Mitchell and Neil, for example, asked on one occasion whether the shape had "only one set of parallel sides" and praised themselves for their capacity to use the affirmative answer to guess that the shape was a trapezoid. On a different occasion, they also asked an overly restrictive question—whether the diagonals were perpendicular

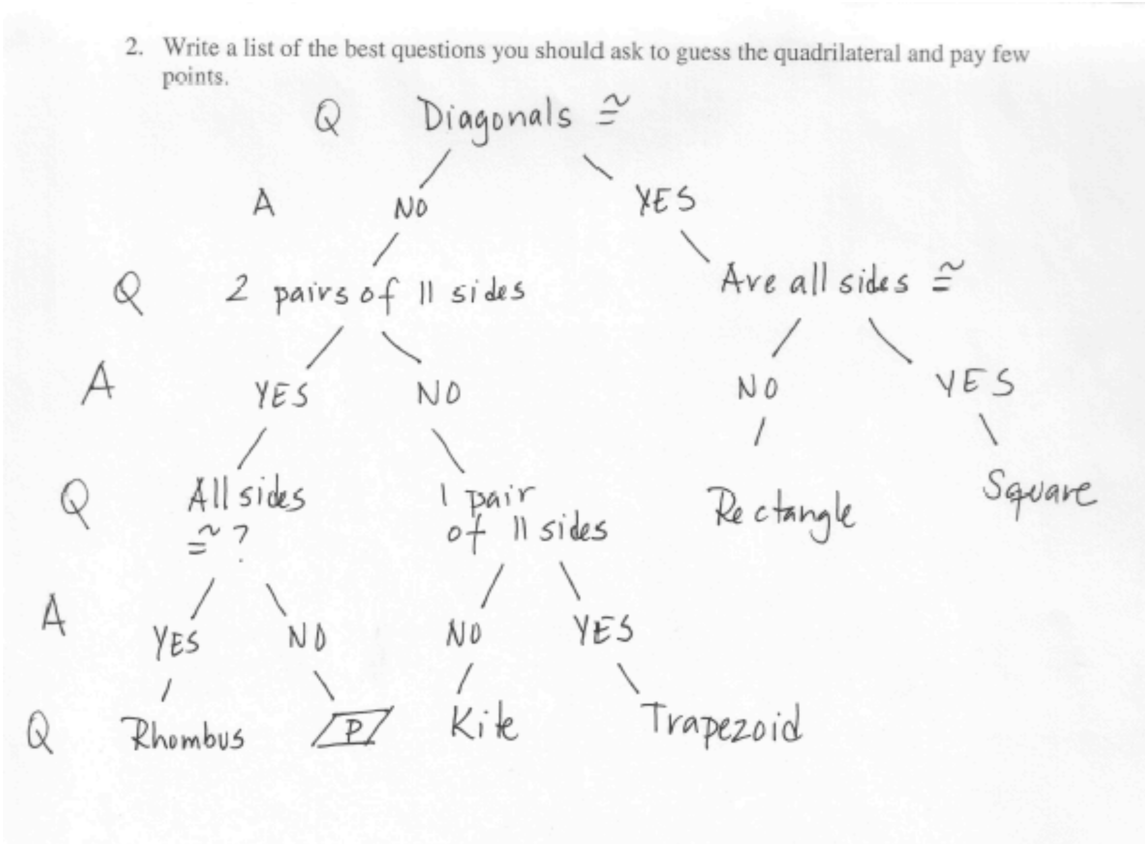

Figure 1. A flowchart for deciding how to play the game

bisectors of each other—and the negative answer kept hidden from them a fact that they might have been able to get and use had they asked a simpler question. Since the unknown shape was a kite, it would have been helpful to ask, for example, whether one of the diagonals was a perpendicular bisector of the other one. So whereas they could see the benefit of getting at the defining property of a shape when they got the expected answer to a complex question, they could also understand how, on average, it would be advantageous to ask simpler questions, whose answers could provide useful information no matter what the answer was.

Some Advantages and Disadvantages of the Game

Clearly the game does not do all one would want in developing meaning for definitions. For example the game is somewhat neutral in regard to whether one should prefer defining special quadrilaterals as a hierarchy (whereby a parallelogram is a trapezoid; see Craine and Rubinstein, 1993) or as a set of disjoint categories (whereby parallelograms are not trapezoids). Mathematically there is little interest in defining these figures as disjoint categories; yet students could successfully play the game even if they did think that special quadrilaterals are disjoint categories.

Additionally, the game promotes the creation of general decision strategies such as lists of questions that will eliminate as many options as possible. Those do not always generate properties that would define a quadrilateral, even though they may be optimal for playing the game. Thus those groups that arrived at one such strategy, like the one shown in Figure 1, could generate a kite as a quadrilateral that does not have equal diagonals and does not have any sides that are parallel. The list of questions works to make a decision (given the available shapes), but the list of answers does not define a kite. Furthermore, a decision strategy based on the flowchart in Figure 1 would compel one toward defining a rhombus as a quadrilateral with all sides congruent and two pairs of parallel sides, but would not make it equally compelling to define it, say, as a quadrilateral with diagonals that are perpendicular bisectors of each other. Moreover, as Mitchell realized in his group, deciding on the order in which question are asked is very important as far as knowing what the answers mean. In that sense, the set of responses to the questions is different than the set of clauses in a mathematical definition—for the latter it is grammatical structure rather than an order that indicates how the information each clause provides will be combined.

But the game was very successful in getting students to think about how much purchase they get with the stipulation of a condition. During the guessing of one figure in Heidi's group, Mitchell commented that he had made a "stupid question" when he had asked if there was a pair of congruent angles. The teacher encouragingly said that the question was in fact a good one. But Mitchell responded, "How much does it rule out?" stressing the point that the quality of the question depends also on how much information it gives at the point of the game in which it is asked. A specific example of how students used this idea in playing is provided again by what happened in Alana's group after they heard that the quadrilateral did not have all angles congruent and they went on with the question of whether the unknown figure had two pairs of congruent opposite sides. On account of having gotten an affirmative answer, Alana suggested that it could be a rectangle, but Madeleine quickly retorted that in that case all angles would be equal. Alana agreed, and she then suggested that the information they had at the time meant that the unknown figure could be "a rhombus or a parallelogram." Similarly, when they decided that the next question would have to ask for the sides, Madeleine added "we know it's not a square," which they used later on to conclude, after finding out that sides were equal, that the figure was a rhombus. Thus students used the information gathered to decide not only what to ask next but also how to interpret the responses to the following questions. They could experience an important quality of a definition: that in saying what something is one is also saying what something is not.

Using a Game like this to Teach Students the Definitions of Figures

Playing the game Guess my Quadrilateral! does not substitute for spelling out definitions and, later on, formulating and proving properties of figures. The game is not even the only thing one should do before spelling out the definitions. The game is a great tool to activate previous knowledge, but it is not just that either. The game provides an important anchor for what it means to stipulate the various conditions that one puts in a definition, what each condition allows and what it rules out. In so doing, the game helps establish why one would want definitions to be succinct rather than verbose, even if one knew (as our students know) that many things are true about those concepts. Further, it helps students realize that alternative definitions could be provided for the same geometric figure: For example, students could be asked to compare the way the flowchart in Figure 1 generates the rhombus with a different set of questions.

In our case, we used the records of how people had played the game as a resource in getting students to formulate mathematical definitions for the shapes. Probably because the game had also activated students' previous knowledge, when, after playing the game we asked students to make up definitions for the shapes, they did include some that were more like descriptions of all that they knew about a figure. Indeed, of the 13 groups, six responded to this task by providing verbose, kitchen-sink descriptions of quadrilaterals immediately after playing the game.

But the experience of having played the game made it possible and meaningful for the teacher to engage students in discussing a question that paved the way to making their definitions succinct: What do you have to know about a shape to make a successful call as to what that shape is? This question does not ask for a definition, but produces one—and can be held against other definitions that students might make—of the "description" or "name and point" nature. As the teacher told one of the classes, even though they might say "Oh, that's important to know," the game had helped them realize that such a property "maybe really [is] not that important and maybe I can say less and still figure out what shape it is." After the class had looked at definitions for the special quadrilaterals only the work of one group exhibited a "definition" that could be more properly called a description. All of this stresses a point that Lakatos (1976) makes with his example of the concept of polyhedron: Definitions for mathematical ideas are shaped considering the theorems that could be proved thereafter.

The teacher also pointed out that definitions might differ. While for one group a particular property might be a part of the definition of the figure, for another group this property could be deduced from the definition of the figure. The game helped people understand why that makes sense—to the extent that a definition really is an efficient tool for intellectual (rather than visual) recognition, students could readily accept that a rhombus might be defined as a quadrilateral with four equal sides or as a quadrilateral whose diagonals are perpendicular bisectors of each other. They also understood that whatever one chose as the definition would condition not only what theorems one could formulate and prove but also what could be used in proving those theorems. Eventually, that seems to be the whole point in our insisting that students know the definitions — not because there is a lot of doubt that they know what words mean but because the idea is to have students understand that knowledge as a system of connected propositions.

The *Principles and Standards for School Mathematics* (NCTM, 2000) proposes as a standard for instructional programs that students be given opportunities to "analyze properties and determine attributes of two- and three-dimensional objects; explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them" (p. 308). Along those lines, more important than knowing the facts of each figure is knowing how those facts are connected and how they can be organized as strong theorems derived from cleverly chosen definitions. Hence, it is not as much knowing the exact official definition that we should strive for as it is engaging students in the mathematical activity of defining—and for this, involving them in generating (or prescribing) figures might be a useful pedagogical tool.

References

Craine, T. & and Rubinstein, R. (1993). A quadrilateral hierarchy to facilitate learning in geometry. *The Mathematics Teacher, 86*, 30–36.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.

Moise, Edwin. (1975). The meaning of Euclidean geometry in school mathematics. *The Mathematics Teacher, 68*, 472–477.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author,

Poincaré, H. (2001). Mathematical definitions and education. In S. J. Gould (Ed.), *The value of science: Essential writings of Henri Poincaré* (pp. 441–459). New York: Modern Library of Science. (Original work published 1914).

[1] One could, however, later on, engage students in activities of generating figures that do not involve already drawn figures but rather require students to also find a model for their prescribed figure. We did not do that in the unit we taught at this time.

[2] Names are pseudonyms.

Appendix A – Student Questionnaire

Quadrilaterals unit - Diagnostic assessment
Name:
Class period:
Date:

I. Check a box if the figure has the property

Property	*Square*	*Rectangle*	*Rhombus*
1. It has two pairs of parallel sides			
2. Its diagonals bisect each other			
3. Its diagonals are congruent			
4. Its diagonals are also angle bisectors			
5. Its diagonals are perpendicular			

II. Always-Sometimes-Never
Write A, S, or N in the Answer box, if you think the statement is <u>A</u>lways true, <u>S</u>ometimes true, or <u>N</u>ever true

Answer	Statement
	A rhombus has equal angles
	A square is a rectangle
	The diagonals of a rhombus make an obtuse angle
	A rhombus is a kite
	A rectangle has congruent diagonals

III. Definition and properties of rectangles

What is a rectangle?

What else do you think is true about the rectangle?

IV. Bisectors of a parallelogram
If you drew the four angle bisectors of a parallelogram, what could you say about the figure they make? Why?