

A Mathematical Microworld to Introduce Students to Probability

Zhonghong Jiang and Walter D. Potter

The objective of this paper is to describe a simulation-oriented computer environment designed to introduce probability and a teaching experiment that evaluates its effectiveness. This learning environment is a mathematical microworld named Chance.

It is a consensual opinion that an understanding of probability and the related area of statistics is essential to being an informed citizen (National Council of Teachers of Mathematics [NCTM], 1989; Shaughnessy, 1992). Some researchers even indicate that perhaps no other branch of the mathematical sciences is as important for all students, college-bound or not, as probability and statistics (Schultz, 1989; Shaughnessy, 1992). Among many attempts to encourage an increased emphasis on probability, the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) recommends including probability in the core curriculum at all levels. There is a wide gap between these ideals and reality, however, because probability is still a weak spot in school mathematics curriculum and instruction. Many researchers (e.g. Kaput, 1990; Konold, 1991) believe that computer technology is well suited to bridge this gap. We agree with them.

According to the *Standards*, “the study of probability in grades 5-8 should not focus on developing formulas or computing the likelihood of events pictured in texts. Students should actively explore situations by experimenting and simulating probability models” (p. 109). “In grades 9-12, students should extend their K-8 experiences with simulations and experimental probability to continue to improve their intuition” (p.171). Therefore, devising and carrying out experiments or simulations to determine probabilities is important for all students. Some basic random devices such as coins, dice, and spinners are indispensable for students to perform experiments while they learn introductory probability concepts. Using physical manipulatives to do probability experiments, however, has its limitations.

Zhonghong Jiang is a doctoral student in the Mathematics Education Department at The University of Georgia. He is interested in the use of technology in mathematics instruction and curriculum development.

Walter D. Potter is an Assistant Professor in the Department of Computer Science and the Graduate Coordinator for the Artificial Intelligence Programs at The University of Georgia. His research interests include intelligent information systems, genetic algorithms, and intelligent tutoring systems.

The limitations of physical manipulatives

Probability phenomena are subject to the law of large numbers, which, intuitively, reflects that if in n identical trials, event A occurs v times, and if n is very large, then v/n should be near the theoretical probability of A . Without a very large number of trials of an experiment, the results will not necessarily convince a student that certain events have the same chance of occurring. An example which shows the importance of a large number of trials was provided by an eight grade student who participated in our study for evaluating the Chance microworld. After he did several coin flipping experiments with Chance, he wrote, “When I flipped a coin 10 times, I got two heads and eight tails. When I flipped it 100 times, I got 59 heads and 41 tails. When I flipped it 1000 times, I got 503 heads and 497 tails.” It is very unrealistic, however, to ask a student to flip a physical coin a very large number (e.g. 1000 or even more) of times. Some teachers may rely on the collective efforts of a whole class of students, but this strategy does not help with small group activity and individual learning.

The results of some experiments are influenced by the methods students use. For instance, a scientist studying coin tossing identified a series of variables that affect the way in which a flipped coin falls (Berkeley, 1961). Those variables include the height of the hand above the table, the angle at which the hand is held, the position of the thumb in relation to the other fingers, the amount of force applied by the thumb to the coin, and whether the coin is heads up or tails up at the start of the flip. The influences of those variables are generally not recognized. If a student is told to put his thumb under the edge of a coin and flip it “in such a way that it turns over in the air at least three or four times” (Berkeley, 1961, p. 5), and another student simply flips a coin without considering how to flip it “correctly,” then the two flipping methods may or may not show mutually consistent results.

Some devices have their own limitations in terms of design or fabrication. For example, a so-called “fair” die is not necessarily guaranteed to be fair. Perhaps the most significant limitation of physical materials is that while doing experiments, students need to record data and translate between different notation systems. These actions “have a tendency to consume all of the student’s cognitive resources even before translation can be carried out, let alone be monitored” (Kaput, 1992, p. 529). Kaput (1992)

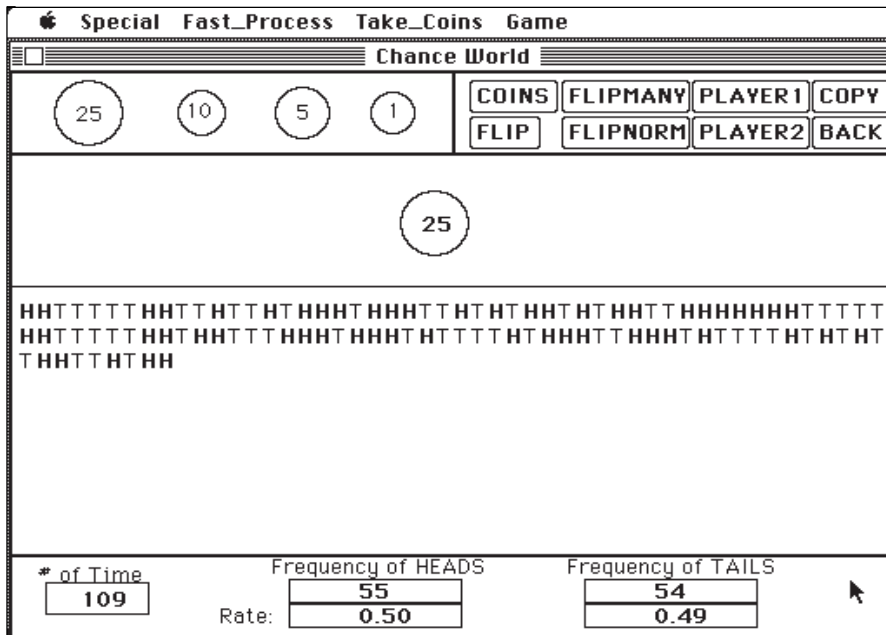


Figure 1: The Coins environment during an experiment. A graphical display, a literal expression, and the numerical records are shown.

also notes that physical materials “are infrequently used except in the earliest grades, and even when they are used, they are seldom part of a sustained learning activity but rather are used to provide a brief illustration of a more abstract concept or operation” (p. 529). All of these limitations, coupled with the demands on teachers’ instructional time, make it impractical to rely solely on physical materials for probability experiments.

Overcoming the limitations of physical manipulatives

These limitations can be overcome by using electronic instantiations of physical materials to build experimental environments. The rapid operation speed of a computer makes realistic a very large number of trials of an experiment. The random number generator of a mature computer programming language, though not perfect (in terms of its cyclic behavior), can be used to simulate various kinds of random processes with high accuracy. Physical materials usually have two different but overlapping functions. One is to provide task and activity structures, and the other is to provide constraint-support (CS) structure (Kaput, 1992). However, research studies (e.g., Thompson & Thompson, 1990) have shown that it is possible to impose a much stronger CS

structure using the computer instantiations of physical materials, rather than the physical materials themselves. This is true especially at the level of coordinating between notation systems or between aspects of situations and sets of notations, because one can transfer to the computer the mechanics of the translation process. The students are thereby freed to focus on the connections between actions on different systems.

Computer simulations and other computer-based activities should be the directions in which probability education is headed. To date, however, only a few studies have been conducted on the effects that computers or computer simulations have on students’ learning about probability. As to existing software packages related to probability, most of them simply display the simulation process or the simulation results. Because the actions students perform on the physical random devices are not reflected in this kind of software, students may find it hard to get a sense of concreteness from the computer simulations. They may feel uncomfortable believing the results.

The microworld approach proposed in *Chance* tries to take full advantage of the power of cybernetic manipulatives, providing a means for students to run a variety of experiments and to collect systematic data. *Chance* can be

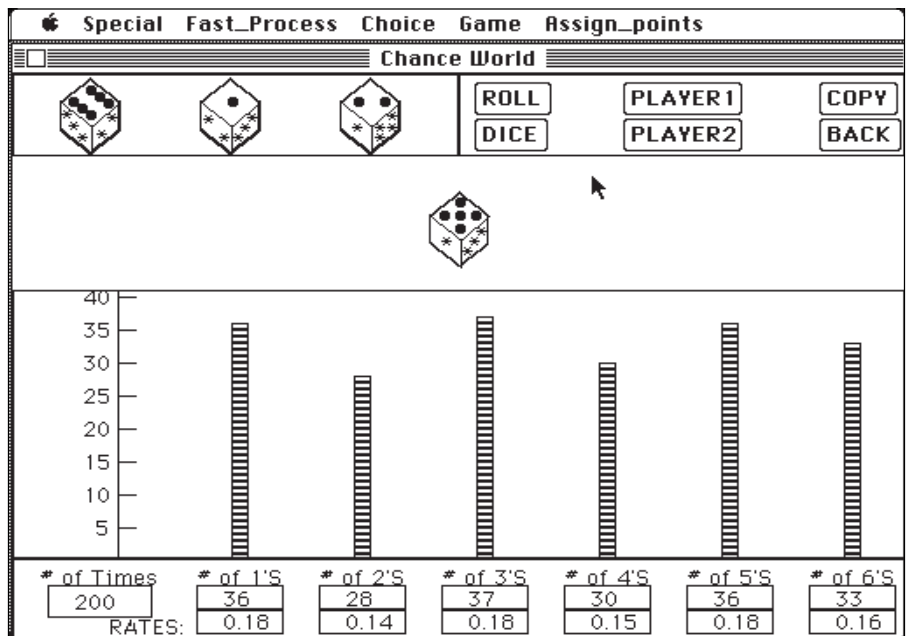


Figure 2: The Dice environment during an experiment. A graphical display, a distribution graph, and the numerical records are shown.

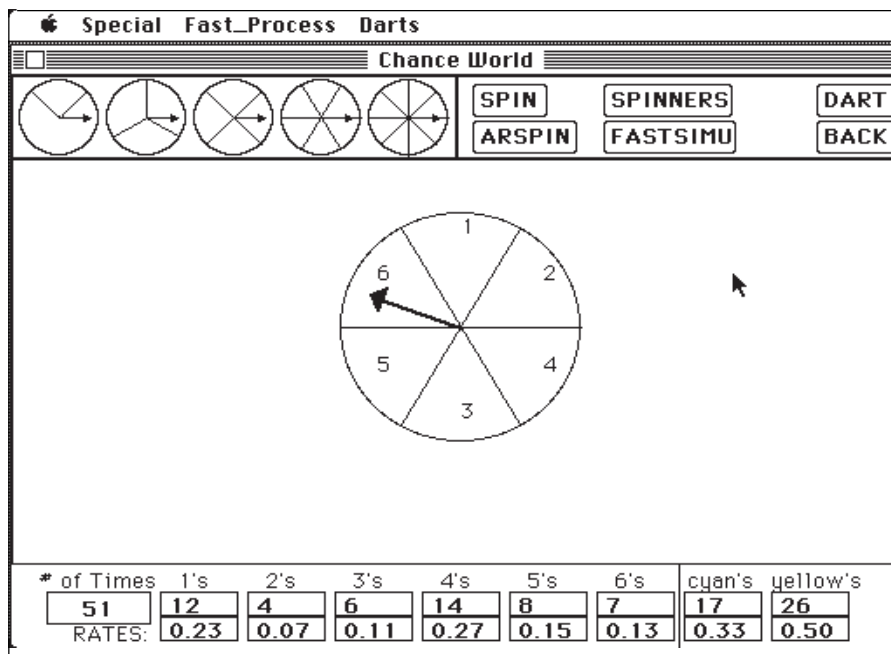


Figure 3. The Spinner environment during an experiment. An arrow spinner is used in this experiment.

thought of as an environment in which students explore their understanding of introductory probability “in the same way that scientists test their conjectures about the way the world works” (Thompson, 1985, p. 201). We believe that this microworld will effectively help students internalize the CS action structure to build their own knowledge structures.

Five sub-environments

Chance has been developed to run on Macintosh computers using Object Logo (Paradigm Software, 1990). To reflect the diversity of various experiments, Chance is designed so that it makes available the major experiments used for learning introductory probability. Five experimental sub-environments comprise Chance: Coins (Figure 1), Dice (Figure 2), Spinners (Figure 3), Thumbtacks (Figure 4), and Marbles (Figure 5). Coins and Dice are the most basic probability experiment environments. Spinners is constructed to study geometric probabilities by providing area models of probabilities. Thumbtacks is built to explore some probabilities that can *only* be determined by experiments. Marbles demonstrates probabilistic situations related to samples and sampling.

The main features of Chance

The most prominent feature of Chance is that it allows actions on its objects, unlike many existing software applications; more specifically, the user can directly manipulate the objects by clicking or dragging the mouse. In other words, Chance changes display notations into action notations. The objects in Chance are the computer instantiations of the physical materials with their current states, such as Head for a coin, or showing a 5 on the top face for a die (Figures 1 & 2). When the user clicks on the die in the middle part of Figure 2, for example, a new object (the die with its new state) appears (Figure 6). This feature provides a vivid simulation of the students’ actions on the random devices

in the real world. Even so, we may still need to encourage students to do initial experiments with physical objects, but the necessity exists only at the beginning for students who have no prior experience with those devices. Also, doing the physical experiments demonstrates the similarity between the physical experiments and their computer-based counterparts so that the students believe that the computer experiments are dependable alternatives to the physical ones. To take full advantage of the computer’s power of rapid operation, Chance has a built-in Fast-

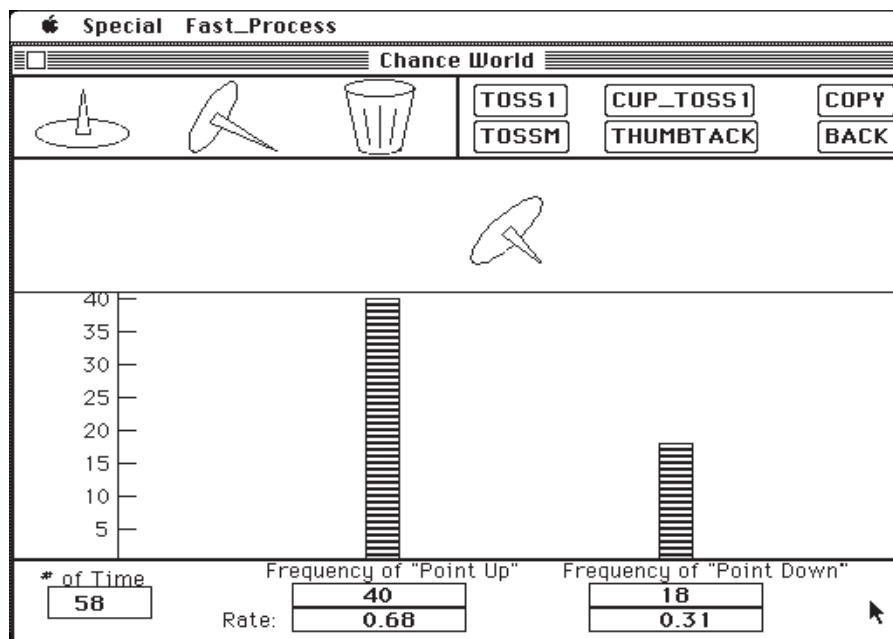


Figure 4. The Thumbtacks environment during an experiment. The difference between the two frequency distributions is apparent.

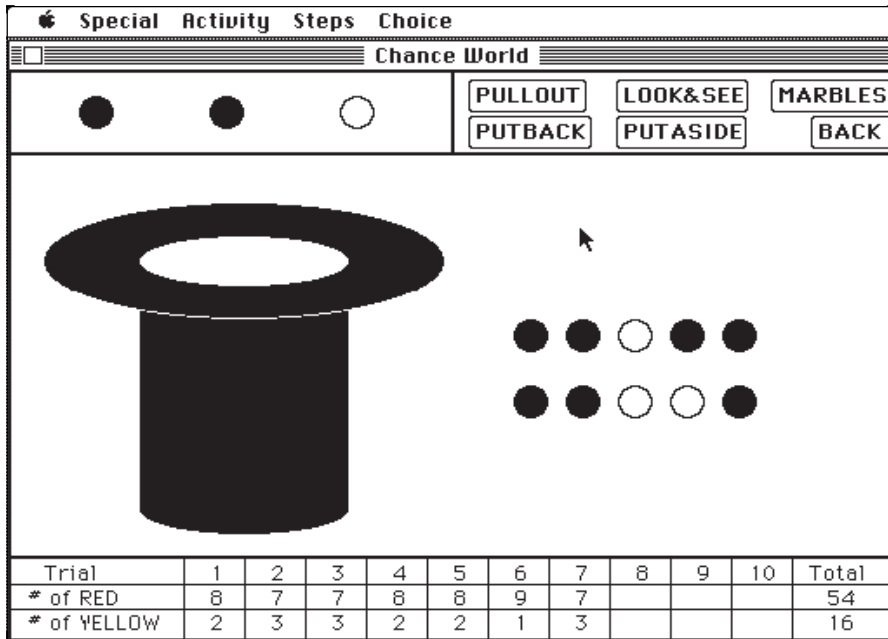


Figure 5: The Marbles environment during an experiment.

Process function. After students manipulate the objects for a while, they may wish to conduct a large number of trials faster. To do so, they can highlight an item under the Fast-Process menu and conduct up to 1000 trials in a very short time period by clicking on the object or one of the objects only once. The fast process can also be repeated and the results accumulated. In addition, the student can stop a fast process at any time. Thus, any number of trials can be realized quickly. As a result, the difficulty of compatibility with the law of large numbers, present with the physical experiments, can be removed completely.

A second important feature of Chance is the use of multiple, linked, and dynamic representations. For the result of each trial of an experiment, several representations can be shown on the same screen: a graphical display, a literal expression or distribution bar expression, and the numerical records (Figures 1 & 2). The graphical display shows the objects with their states, which keep changing along with the user's actions. The literal expression keeps track of the states resulting from those actions. This expression is necessary not only because it records the whole history of state changes during the process of the experiment, but also because it can display all the possible outcomes of a particular experiment and can reflect the characteristics of randomness.

An example of the latter is in the situation of flipping a coin where a run

occurs when the same result happens two or more times. The literal expression can provide the information of the number of runs, the average length of a run of Heads (or Tails) and the average length of all runs (Figure 1). In comparison to literal expressions, the distribution graph (especially the dynamic growing process of the bars along with the user's actions), is more intuitive and more informative to students. It is relatively easier for students to accept the notions of equally likely or not equally likely when they observe the growing bars which frequently have almost the same height (Figure 7) versus those which often have apparently different heights (Figure 4). The numerical records are the results of the translation from the above notation

systems to the numeration system. For the purposes of observations, the numerical results are automatically accumulated as the experiment continues and one can see the accumulation process (Figures 2 & 6). The linking among all the notation systems mentioned above is reflected in the following fact: when an action causing a state change is taken on the graphical display, the new state(s) is expressed literally (or the corresponding bar grows), and each update of the numerical expressions is shown simultaneously. By off-loading on the computer the mechanics of the translation process and coordinating among different notation systems this way, Chance can help students

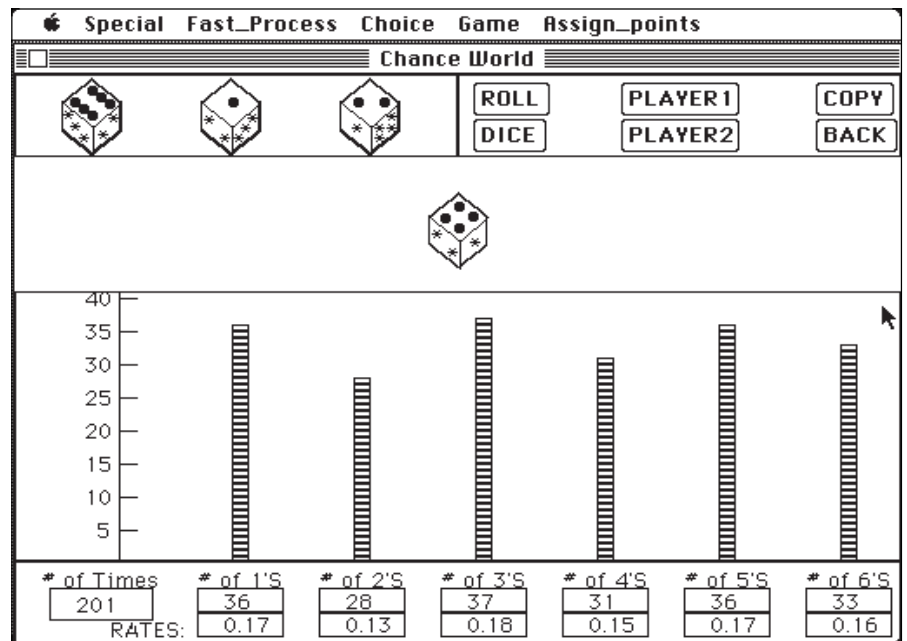


Figure 6: The screen that resulted from clicking on the die (showing a 5 on the top face) in Figure 2.

concentrate on the connections between, as well as interpretations of, the observed phenomena.

The third feature of Chance is its built-in constraint-support structure (CS structure). One significant feature of this structure is its auto-scaling function in distribution graphing. The vertical axis is automatically scaled so that the changing bar extends from the bottom to the top of the allocated region without being limited by the number of experiment trials (Figures 4 & 7). More constraints or supports on actions can be seen in that the system often responds to an input with an admonition or a comment. In performing a particular experiment, it is possible for a student to do something incompatible with that situation such

as clicking an inappropriate key or unintentionally bringing in extra objects. In such a case, an admonition is always given. An example is in the Thumbtacks sub-environment. In an experiment of tossing a cup, if a user mistakenly clicks the **TossM** button, then the system will say, "Sorry, this is in the process of tossing one object. Please use the **Cup_Toss1** button to continue." An example of supportive comments is found in the dice rolling environment, where a record can be set for the most rolls of two dice without getting doubles (a "double" means both dice show the same number). The system gives appropriate comments such as "Not a double, keep rolling!", "Good

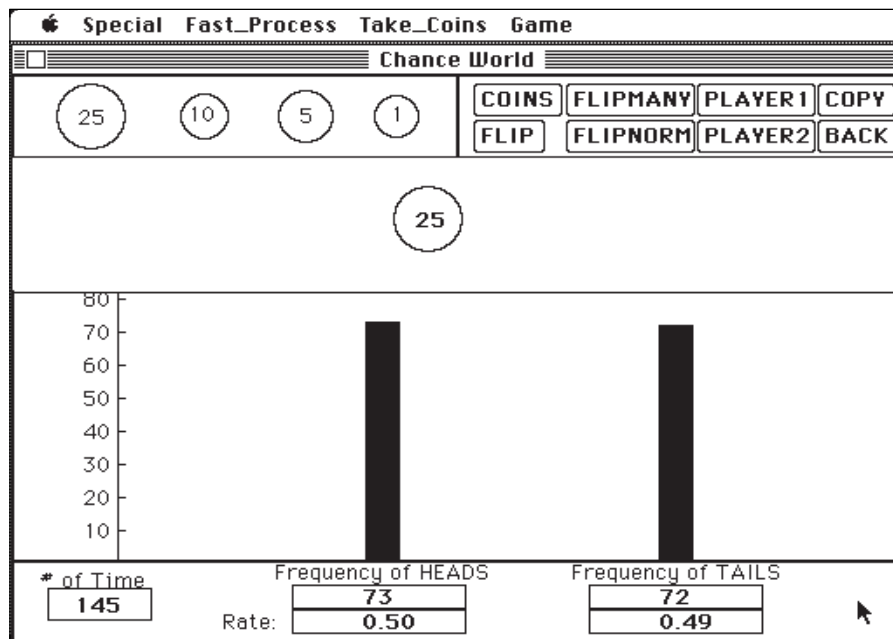


Figure 7: The two distribution bars are almost the same height.

record! Keep rolling!", "Very good!", and "Excellent!" (Figure 8). These supports and constraints help overcome cognitive overload problems, thus freeing the students to focus their efforts on conceptual understanding.

The fourth feature of Chance is that the experiment environments are designed to be as flexible as possible so that

- many experiment cases can be involved;
- many real world situations can be reflected; and
- students are allowed to play games within the environments.

In the coin flipping situation, for example, the user can choose to flip any number of coins. Flipping one coin any number of times allows the student to predict and verify the probability of various outcomes, and to examine the number and lengths of runs (Figure 1). Flipping two or more coins any number of times allows for exploration of "equally likely" events, compound events (Figure 9), and the effect of different sample sizes.

Furthermore, this flexibility allows students to model many real world probability situations. For example, the Coin environment can be used to model the probability that a family of four children will consist of two girls and two boys. Another real world situation that can be modeled involves the effect of hospital size on the number of days (over any specified period of time) upon which at least 60% of children born are boys.

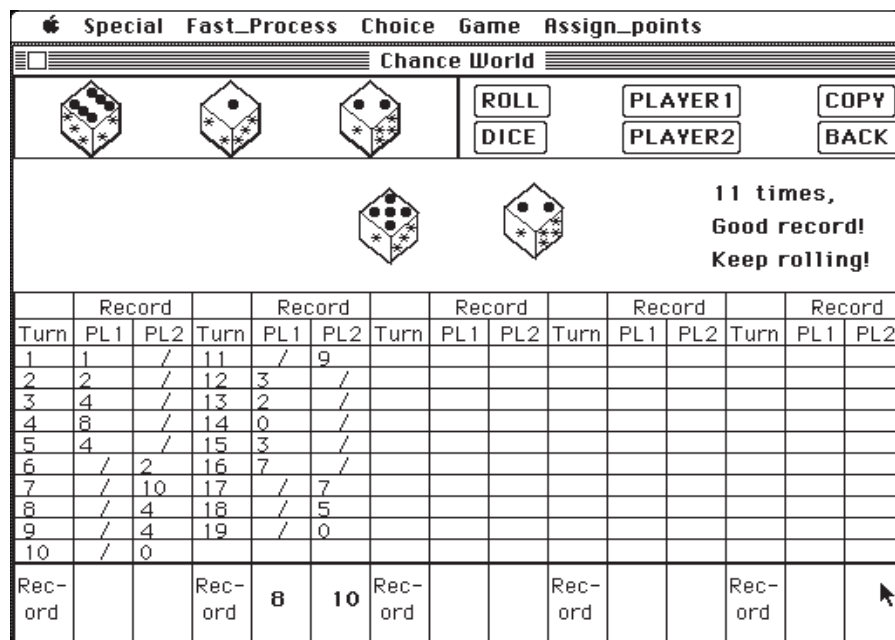


Figure 8: The Dice sub-environment responds to an input with a comment.

Finally, students can play many games within this environment and discuss the fairness or unfairness of these games. One of the games available is the following: Two players each flip a coin. Player 1 gets one point if there is a match. Since there are two ways to get a match, player 2 will be given two points if there is no match. If students think the game is unfair, they can try to change the rules to make it fair. The modified game can then be played. Due to its flexibility, Chance can be used at a number of grade levels. These microworlds can provide a range of experiences from introductory probability experiments to solving by modeling problem situations and doing experiments.

The fifth feature of Chance involves asking students to make predictions before any experiment starts. This reflects one step in the following general format for the student's exploratory activities: posing a question or a problem, making predictions, doing the related experiment and recording the data, comparing the data with the predictions, keeping or revising the predictions, and giving interpretations. The purpose of using this format is to encourage students to participate more actively in the experiments, so as to get a better understanding of the relationship between the numerical expression of a probability and the events that give rise to these numbers. Throughout the instructional process involving predicting and comparing outcomes, students will be "constantly placed in a position of having to reconcile the dissonance between their probability misconceptions and their empirical observations" (Shaughnessy, 1992, p. 482).

With all the above features, Chance functions the same as the physical experimental environment in the sense that it provides the computer instantiations of the random devices that can be manipulated. However, Chance is superior to the physical experimental environment in the sense that it overcomes the inherent limitations of the latter.

The design of the interface

As depicted in the accompanying figures, for each sub-environment, the screen is generally divided into five regions: 1) the menu region, 2) the button region, 3) the toy region where as many toys (coins, dice, etc.) as needed can be taken, 4) the experiment region which is the central part of the screen, and 5) the data region, where the results of the experiment trials are accumulated and recorded. Depending on the needs of the sub-environment, the experiment region is sometimes further divided into two parts: the upper part depicting the visualization of the microworld's objects with their states, and the lower part displaying the dynamic changing process of the literal representation or the distribution graph of the experiment results (or displaying a table tracing the scoring process in a game situation). For students to enter each particular sub-environment easily, a starting screen (Figure 10) is provided.

To minimize the student's cognitive effort and promote better interaction between the student and the system, complex syntax is avoided. Instead, labeled buttons were constructed, and new pull-down menus were built to replace the standard menu bar. Also prompts are given whenever needed. Clicking the buttons allows students to make their action choices. The new menu items assist students in making such choices by either setting values for various parameters or providing a particular setting (e.g., for playing games). The button labels, the menu titles, and the menu item names are chosen by either the natural language conventions or the conventions established in mathematics.

A teaching experiment

In order to evaluate the performance of Chance, a teaching experiment was designed and implemented. The purpose of the teaching experiment was to see whether Chance can

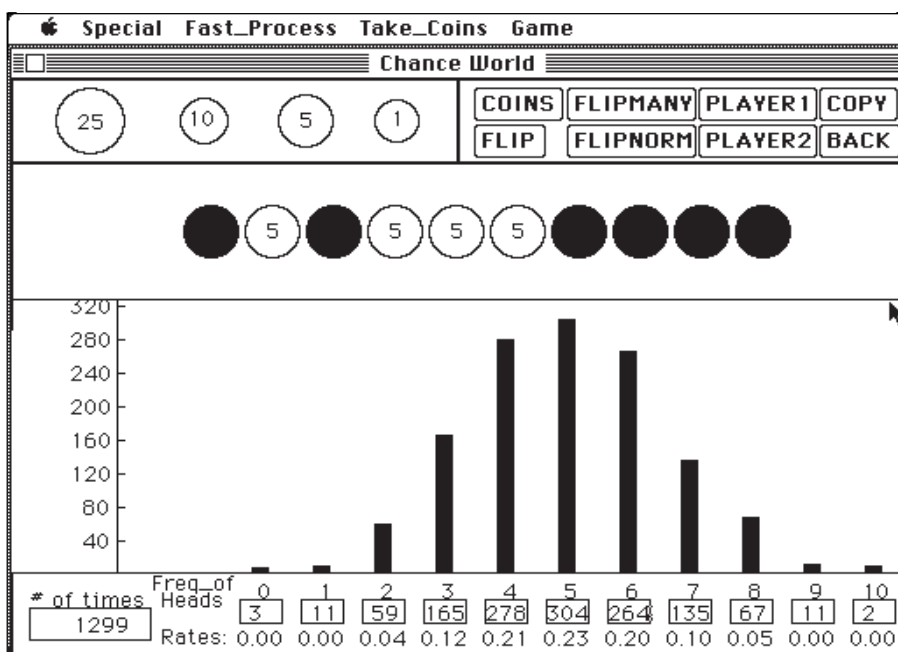


Figure 9: An example from the Coins sub-environment exploring the probabilities of compound events and showing a distribution that is approximately normal.

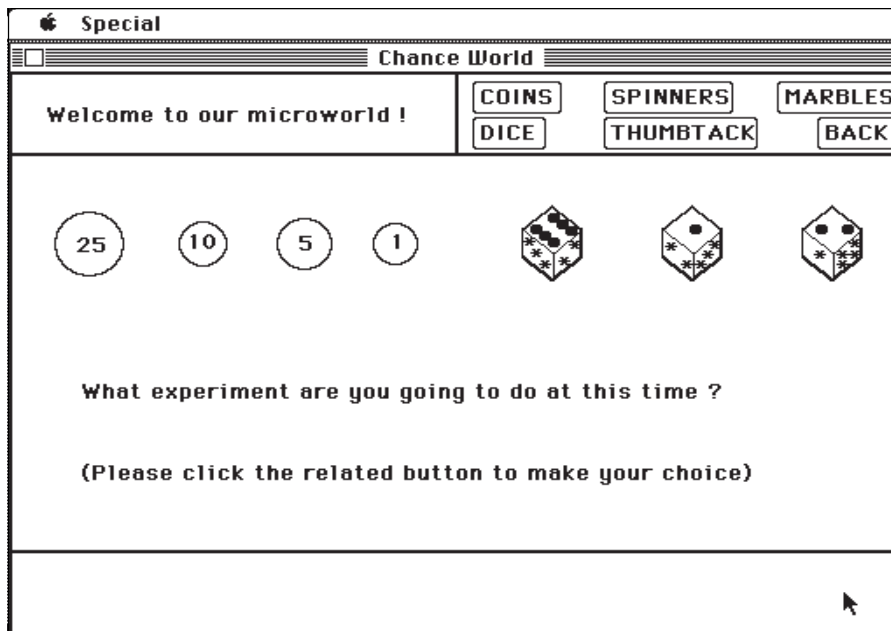


Figure 10. The starting screen.

help students achieve conceptual understanding of probability. Therefore, rather than obtaining limited information from a large sample of subjects, more detailed information was obtained from a smaller sample. Three middle school students (a fifth, sixth, and eighth grader) and one high school student (a junior) participated in the study. Each of them worked with the investigator on a computer for two sessions per week for about five weeks. The teaching experiment consisted of three phases.

The first phase contained two sessions, arranged for pretest and free play. The performance of the subjects on the pretest showed that they had a sound understanding of prerequisite concepts such as fractions, decimals, percents, and ratios. However, they exhibited misconceptions about probability, using judgmental heuristics to make their probabilistic decisions (see Shaughnessy, 1992). The purpose of free play was to allow the subjects to become familiar with the software.

The second phase consisted of seven instructional sessions. During this phase, key concepts such as events, the likelihood of such events, randomness, equally likely events, independent events, dependent events, and compound events were discussed in the experimental environments. The problem-solving method of modeling problem situations by devising and carrying out experiments was emphasized.

The third phase was a posttest (one session). The posttest assessed students on their knowledge of both the basic concepts and their ability to solve problems by simulation and modeling. All the four subjects did very well on it; each of them correctly answered all three questions on basic concepts and performed better on the problem solving items.

Some observed potentials of Chance

When reviewing the results of the teaching experiment, much evidence was found that Chance can effectively enhance students' learning and understanding of probability concepts. The following are some observed outcomes of using Chance.

Stimulating students' interest in learning probability

All four subjects were strongly attracted to the learning activities organized in the Chance environment. One of them said, "I can quickly finish an experiment to check my idea. I like it. It's not boring at all." Another child was even more excited, "It is really interesting. It attracted all my attention."

The various games that can be played, the graphical display that changes along with the user's actions, the distribution bars that grow dynamically, as well as the convenience provided by the fast process feature and the auto-scaling function, were all found to stimulate the subjects' interest in learning probability.

Helping students overcome their learning difficulties

The multiple representation feature of Chance frees students from recording data and translating between notation systems. It can also help students overcome their learning difficulties. An example of this aspect was noted when the subjects extended their experience of flipping one coin, two coins, and three coins to the situation of flipping four coins. After smoothly completing the activities associated with flipping three or fewer coins, the two younger children became confused when they were asked to predict how many ways the four coins could land. Both of them knew that they should look for a pattern from previous experiences. However, the patterns they found were either incorrect or incomplete. Both of them concentrated on the numbers two, four and eight that were obtained previously, but they neglected the close relationship between the numbers and the situations where they were obtained. To help the two children recognize the importance of looking for patterns in context, the investigator suggested they use the Flip button to do the experiments for one, two and three coins again and pay very close attention to the relationship between flipping n coins and flipping $n-1$ coins ($n = 2, 3$). The tree structures displayed on the screen promoted the children's understanding. The sixth grader said almost immediately, "Oh, I got it." He quickly drew the same tree diagram displayed for three

coins, and extended it in a somewhat systematic way to form a tree diagram for four coins. In a short while, the tree diagram also made sense to the fifth grader. Both of them understood that the number of ways for n coins to land is 2 times the number of ways for $n-1$ coins to land (when n is greater than one) and that the 2 represents the two possibilities for the n th coin (H or T). The two children continued to explore the pattern and constructed the formula: the number of ways for n coins to land is 2^n .

Helping students change their misconceptions about probability

Chance provides quick feedback for students to evaluate their predictions. In the teaching experiment, we found that this feature can help students change their misconceptions. For example, most of the subjects thought the measure of uncertainty does not vary as sample size changes, as reflected in their solutions to the hospital problem. To counter this misconception, students were asked to use the FlipMany button to simulate ten trials of flipping 5, 10, and 100 coins. They were to keep records, give conclusions, and reconsider the hospital problem. The results of their experiments are displayed in Figure 11.

This table was very informative, showing a sharp contradiction to their predictions of the effects of the sample size. The students quickly recognized that their solutions to the hospital problem were wrong and recognized that sample size does matter in measuring uncertainty.

One point that should be emphasized is that helping students change their misconceptions by performing experiments is much harder in the physical environment. For example, it is unreasonable to ask a student to flip 100 physical coins simultaneously and record the results.

Saving instructional time

The teaching experiment has verified that instructional time can be saved by using Chance. Several experiments with large numbers of trials were completed in one teaching episode which would have been impossible using physical devices. Students' performance on the posttest showed that more teaching had been done in the same time frame than could have been done in the physical experimental environment.

The shortcomings of Chance

Several shortcomings of Chance were revealed by the teaching experiment. First, though Chance makes available the main experiments in teaching introductory probability, some interesting experiments cannot be conducted in it. For example, one cannot flip a coin and spin a spinner in the same experiment. It is also impossible to roll an icosahedral die (twenty faces with the digits 0-9 appearing twice). Such experiments are important, and sometimes even necessary. Although Chance can help students change misconceptions, misconceptions are not directly addressed. Assistance is needed to help those students who either do not know how to explore ideas or do not know how to generalize from examples. To assist these students, misconceptions must be diagnosed and remediated (Thompson, 1987). Work in this direction has not yet been started.

Future work on Chance

Future work on this microworld will take two directions. One direction is to increase the flexibility of the

Trial	Flip 5 coins		Flip 10 coins		Flip 100 coins	
	# of Heads	# of Tails	# of Heads	# of Tails	# of Heads	# of Tails
1	2	3	6	4	49	51
2	2	3	7	3	51	49
3	1	4	5	5	57	43
4	3	2	4	6	52	48
5	3	2	6	4	51	49
6	3	2	5	5	56	44
7	4	1	4	6	51	49
8	3	2	4	6	59	41
9	4	1	5	5	56	44
10	1	4	4	6	47	53
# of trials at which at least 60 % Heads were obtained						
6		3		0		

Figure 11: Table showing students' results of ten trials of flipping 5, 10, and 100 coins.

system by making more experiments available in the five existing sub-environments. In addition, new sub-environments should be created according to the needs of curriculum and instruction. In order for high school students to "extend their K-8 experiences with simulations and experimental probability to continue to improve their intuition" (NCTM, 1989, p. 171), experiments that involve more advanced concepts and more advanced simulation techniques should be seriously

considered and included. Some of the more advanced topics which might be included are the concept of a random variable that can be applied to generate and interpret binomial, uniform, normal and chi square probability distributions; the concept of conditional probability; the Markov chain, which can be used to predict weather; and the more systematic study of the Monte Carlo procedure, which can be used to simulate almost any probability or expected value problem. The other direction is to make Chance more intelligent, so that the system can pose more problems, further evaluate responses, and give more detailed and appropriate feedback. These features would be most helpful for the students who need more guidance in their explorations. For most students, their own active exploration and construction will lead them into encounters with many ideas about probability, which is the primary purpose of designing a mathematical microworld.

Note

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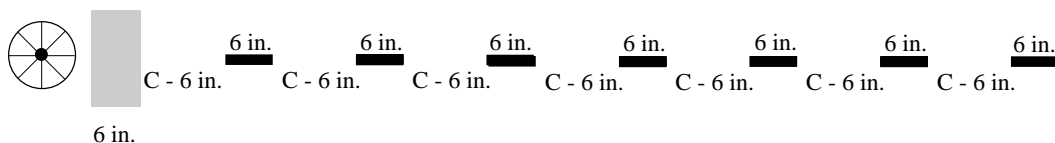
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Problem Solutions

Bicycle Tracks

The only factor which affects the pattern that will be made by the tires is the circumference of the tires. Let C be the circumference of the tire in inches. Assuming that the circumference of the front tire is the same as that of the back tire, the following pattern will be made:



Fred and Frank

Although it is possible to attain an algebraic solution to this problem using the traditional formula involving distance, rate, and time, an equally valid solution can be obtained using quantitative reasoning. If Frank runs half of the time, he will cover more than half of the distance in that time. Thus, he will finish first.

To further convince yourself that this is true, try an example using specific values. For example, let the distance be 100 miles, let the walking rate be 5 miles per hour, and let the running rate be 10 miles per hour. Then Fred runs for 5 hours and walk for 10 hours, for a total of 15 hours to cover the distance. Frank runs and walks for approximately 6.67 hours, for a total of approximately 13.34 hours. Thus, Frank finishes first.